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Research Article

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Superstability of functional equations related to spherical functions

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Abstract: In this paper we prove stability-type theorems for functional equations related to spherical functions. Our proofs are based on superstability-type methods and on the method of invariant means.

Keywords: Spherical function, Stability

MSC: 39B82, 39B52, 43A90

1 Introduction

In this paper $\mathbb{C}$ denotes the set of complex numbers. We suppose that G is a topological group and K is a compact topological group of continuous automorphisms of G. Hence, as a group K is a subgroup of the group of $\operatorname{Aut}(G)$ of all continuous automorphisms of G. We also assume that the mapping $k \mapsto k(x)$ from K into G is continuous for each x in G. The normed Haar measure on K is denoted by m_K. Hence m_K is right and left invariant and $m_K(K) = 1$. We shall consider the functional equation

$$\int_K f(xk(y))\,dm_K(k) = g(x)h(y) + p(y)\,, \tag{1}$$

where $f, g, h, p : G \to \mathbb{C}$ are continuous functions, and f is non-identically zero. Important special cases are

$$\int_K f(xk(y))\,dm_K(k) = f(x) + f(y) \tag{2}$$

corresponding to the case $h = 1,\ f = g = p$, and

$$\int_K f(xk(y))\,dm_K(k) = f(x)f(y) \tag{3}$$

corresponding to the case $p = 0,\ f = g = h$. Nonzero solutions f of the latter equation are called *generalized K-spherical functions*. We note that if f is a bounded solution of (3), then we call it a *K-spherical function*. For K-spherical functions see [1]. Functional equations related to spherical functions have been studied in [2–4]. In the case G is a discrete group and $K = \{id_G\}$ then (1) reduces to

$$f(xy) = g(x)h(y) + p(y)$$

which is a Levi–Civitá–type functional equation and its stability was studied on hypergroups in [5]. The stability of sine and cosine functional equations was investigated in [6].

In this paper we study stability properties of functional equations of type (1). The ideas are similar to those in [5–7].

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2 Superstability of the functional equation (1) when $p = 0$

The following lemma is crucial.

Lemma 1. *Let $f : G \to \mathbb{C}$ be continuous, then we have*

$$\int_K \int_K f(xk(y)l(z))\,dm_K(k)\,dm_K(l) = \int_K \int_K f(xk(yl(z)))\,dm_K(k)\,dm_K(l)$$

for each x, y, z in G.

Proof. We apply Fubini's Theorem and the invariance of m_K to get

$$\int_K \int_K f(xk(y)l(z))\,dm_K(k)\,dm_K(l) = \int_K \int_K f(xk(y)l(z))\,dm_K(l)\,dm_K(k) =$$

$$\int_K \int_K f(xk(y)(kl)(z))\,dm_K(k)\,dm_K(l) = \int_K \int_K f(xk(yl(z)))\,dm_K(k)\,dm_K(l).$$ □

The next theorem is about the superstability of the functional equation of type (1) where f and g are equal and $p = 0$.

Theorem 2. *Suppose that $f, g : G \to \mathbb{C}$ are continuous functions such that the function*

$$x \mapsto \int_K f(xk(y))\,dm_K(k) - f(x)g(y)$$

is bounded on G for each y in G. Then either f is bounded or g is a generalized K-spherical function.

Proof. We let

$$F(x, y) = \int_K f(xk(y))\,dm_K(k) - f(x)g(y)$$

for each x, y in G. Then $F : G \times G \to \mathbb{C}$ is continuous and it satisfies $|F(x, y)| \leq A(y)$ with some function $A : G \to \mathbb{C}$ for each x, y in G.

Substituting $yl(z)$ for y and using the fact that $l \mapsto F(x, yl(z))$ is continuous, hence integrable on K, we have, by Lemma 1

$$\int_K \int_K f(xk(y)l(z))\,dm_K(k)\,dm_K(l) - f(x) \int_K g(yl(z))\,dm_K(l) = \tag{4}$$

$$\int_K \int_K f(xk(yl(z)))\,dm_K(k)\,dm_K(l) - f(x) \int_K g(yl(z))\,dm_K(l) = \int_K F(x, yl(z))\,dm_K(l).$$

On the other hand, substituting $xk(y)$ for x and z for y we obtain

$$\int_K \int_K f(xk(y)l(z))\,dm_K(k)\,dm_K(l) - g(z) \int_K f(xk(y))\,dm_K(k) = \int_K F(xk(y), z)\,dm_K(k). \tag{5}$$

Moreover, we have

$$g(z) \int_K f(xk(y))\,dm_K(k) - f(x)g(y)g(z) = g(z)F(x, y)$$

which implies, together with (5)

$$\int_K \int_K f(xk(y)l(z))\,dm_K(k)\,dm_K(l) - f(x)g(y)g(z) = \int_K F(xk(y), z)\,dm_K(k) + g(z)F(x, y). \tag{6}$$

Now, from (4) and (6) we derive

$$f(x)\Big(\int_K g(yl(z))dm_K(l) - g(y)g(z)\Big) = -\int_K F(x, yl(z))dm_K(l) + \int_K F(xk(y), z)dm_K(k) + g(z)F(x, y)$$

for each x, y, z in G. Obviously, the right hand side, as a function of x, is bounded on G. Hence, if f is unbounded, then we must have

$$\int_K g(yl(z))dm_K(l) = g(y)g(z)$$

for each y, z in G, which was to be proved. □

As a consequence we obtain the superstability of the functional equation of K-spherical functions.

Corollary 3. *Suppose that $f : G \to \mathbb{C}$ is a continuous function such that the function*

$$x \mapsto \int_K f(xk(y))dm_K(k) - f(x)f(y)$$

is bounded on G for each y in G. Then either f is bounded or it is a generalized K-spherical function.

Now we are in the position to prove the general superstability-type result for equation (1) in the case $p = 0$.

Theorem 4. *Suppose that $f, g, h : G \to \mathbb{C}$ are continuous functions such that h is nonzero, and the function*

$$x \mapsto \int_K f(xk(y))dm_K(k) - g(x)h(y)$$

is bounded on G for each y in G. Then either g is bounded or $h(e) \neq 0$ and $h/h(e)$ is a generalized K-spherical function.

Proof. Suppose that $h(e) = 0$, where e is the identity of G. Then we have

$$\int_K f(xk(e))dm_K(k) - g(x)h(e) = f(x),$$

it follows that f is bounded, which implies immediately that the function $x \mapsto g(x)h(y)$ is bounded for each y, too. As $h \neq 0$ we infer that g is bounded. This means that we may assume that $h(e) \neq 0$. In this case, obviously, we may replace h by $h/h(e)$, that is we assume $h(e) = 1$.

We use similar ideas like above. We introduce the function

$$F(x, y) = \int_K f(xk(y))dm_K(k) - g(x)h(y)$$

for each x, y in G, then $F : G \times G \to \mathbb{C}$ is continuous, and it satisfies

$$|F(x, y)| \leq A(y)$$

for each x, y in G with some function $A : G \to \mathbb{C}$. We have then

$$\int_K \int_K f(xk(y)l(z))dm_K(k)dm_K(l) - h(z)\int_K g(xk(y))dm_K(k) = \int_K F(xk(y), z)dm_K(k) \tag{7}$$

and

$$\int_K \int_K f(xk(y)l(z))dm_K(k)dm_K(l) - g(x)\int_K h(yl(z))dm_K(l) = \int_K F(x, yl(z))dm_K(l) \tag{8}$$

for each x, y, z in G and k, l in K. It follows

$$h(z)\int\limits_K g(xk(y))dm_K(k) - g(x)\int\limits_K h(yl(z))dm_K(l) = \int\limits_K [F(x, yk(z)) - F(xk(y), z)]dm_K(k)$$

for each x, y, z in G. Substituting $z = e$ and using $h(e) = 1$ we obtain

$$\int\limits_K g(xk(y))dm_K(k) - g(x)h(y) = F(x, y) - \int\limits_K F(xk(y), e)dm_K(k)\,,$$

and

$$h(z)\int\limits_K g(xk(y)dm_K(k) - g(x)h(y)h(z) = h(z)[F(x, y) - \int\limits_K F(xk(y), e)dm_K(k)]\,.$$

Adding to (7) we have

$$\int\limits_K\int\limits_K f(xk(y)l(z))dm_K(k)dm_K(l) - g(x)h(y)h(z) = \tag{9}$$

$$\int\limits_K F(xk(y), z)dm_K(k) + h(z)[F(x, y) - \int\limits_K F(xk(y), e)dm_K(k)].$$

Finally, we subtract (8) from (9) to get

$$g(x)\Big(\int\limits_K h(yl(z))dm_K(l) - h(y)h(z)\Big) = \int\limits_K F(xk(y), z)dm_K(k) +$$

$$h(z)[F(x, y) - \int\limits_K F(xk(y), e)dm_K(k)] - \int\limits_K F(x, yk(z))dm_K(k)\,,$$

and the right hand side is a bounded function of x. Hence if g is unbounded, then we must have

$$\int\limits_K h(yl(z))dm_K(l) = h(y)h(z)$$

for each y, z in G, which is our statement. □

3 Stability of the functional equation (1)

If in the functional equation (1) we have $p \neq 0$, then the equation has some "additive character", too, as it includes equation (2) if $h = 1$. Hence we cannot expect a purely superstability result, which is a common feature of multiplicative-type equations. On the other hand, in the case of additive-type equations our experience shows that invariant means can be utilized. This is illustrated in the following general result.

Theorem 5. *Suppose that G is an amenable group, K is finite and let f, g, h, p be continuous functions with f and h unbounded. Then the function*

$$(x, y) \mapsto \int\limits_K f(xk(y))dm_K(k) - g(x)h(y) - p(y)$$

is bounded if and only if we have

$$\begin{aligned} f(x) &= h(e)[\varphi(x) + \psi(x)] + b_1(x) \\ g(x) &= \varphi(x) + \psi(x) \\ h(x) &= h(e)\omega(x) \end{aligned}$$

$$p(x) = h(e)\varphi(x) + b_2(x)$$

where $\omega : G \to \mathbb{C}$ is a generalized K-spherical function, $b_1, b_2 : G \to \mathbb{C}$ are bounded functions, $h(e)$ is a nonzero complex number, $\varphi : G \to \mathbb{C}$ is a function satisfying

$$\int_K \varphi(xk(y))dm_K(k) = \varphi(x)\omega(y) + \varphi(y) \tag{10}$$

and $\psi : G \to \mathbb{C}$ is a function satisfying

$$\int_K \psi(xk(y))dm_K(k) = \psi(x)\omega(y) \tag{11}$$

for each x, y in G.

Proof. As f is unbounded, hence g is unbounded, too, and the function

$$x \mapsto \int_K f(xk(y))dm_K(k) - g(x)h(y)$$

is bounded for every fixed y in G. By Theorem 4, it follows that $h = c\,\omega$, where $c = h(e) \neq 0$, and ω is a generalized K-spherical function on G. Replacing h by $h/h(e)$ we may suppose that $h(e) = 1$. Putting $y = e$ in the condition we have that $f - g$ is bounded. Let M be a right invariant mean on G and we define

$$\varphi(y) = M_x[\int_K g(xk(y))dm_K(k) - g(x)\omega(y)]$$

for each y in G. Here M_x means that the mean M is applied to the expression in the bracket as a function of x while y is kept fixed. Then, since ω is a generalized K-spherical function, we have

$$\int_K \varphi(yl(z))dm_K(l) - \varphi(y)\omega(z) - \varphi(z) =$$

$$\int_K M_x[\int_K g(xk(yl(z))dm_K(k) - g(x)\omega(yl(z))]dm_K(l) -$$

$$\omega(z)M_x[\int_K g(xk(y))dm_K(k) - g(x)\omega(y)] - M_x[\int_K g(xk(z))dm_K(k) - g(x)\omega(z)] =$$

$$\int_K M_x[\int_K g(xk(y)l(z))dm_K(l) - g(xk(y))\omega(z)]dm_K(k) -$$

$$\int_K M_x[\int_K g(xl(z))dm_K(l) - g(x)\omega(z)]dm_K(k) = 0,$$

by Lemma 1 and by the right invariance of the mean M. Now we obtain

$$\varphi(y) - p(y) = M_x[\int_K g(xk(y))dm_K(k) - g(x)\omega(y) - l(y)] =$$

$$M_x[\int_K f(xk(y))dm_K(k) - g(x)\omega(y) - l(y)] + M_x[\int_K (g(xk(y)) - f(xk(y))dm_K(k)]$$

and here both terms are bounded. It follows that $l - \varphi$ is bounded.

As $f - g$ is bounded we have

$$(x, y) \mapsto \int_K f(xk(y))dm_K(k) - \int_K g(xk(y))dm_K(k)$$

is bounded, hence we have that the function

$$(x, y) \mapsto \int_K g(xk(y))dm_K(k) - g(x)\omega(y) - \varphi(y)$$

is bounded, too. We let

$$|\int_K g(xk(y))dm_K(k) - g(x)\omega(y) - \varphi(y)| \le L$$

for each x, y in G with some constant L. It follows

$$|\int_K \int_K g(xl(y)k(z))dm_K(k)dm_K(l) - \omega(z) \int_K g(xl(y))dm_K(l) - \varphi(z)| \le L$$

and

$$|\int_K \int_K g(xl(yk(z))dm_K(l)dm_K(k) - g(x) \int_K \omega(yk(z))dm_K(k) - \int_K \varphi(yk(z))dm_K(k)| \le L.$$

From these two inequalities, by (10) and the property of ω, we infer

$$|\omega(z)\Big(\int_K g(xl(y))dm_K(l) - g(x)\omega(y) - \varphi(y)\Big)| \le 2L$$

for each x, y, z in G. As $\omega = h$ is unbounded it follows that we have

$$\int_K g(xl(y))dm_K(l) = g(x)\omega(y) + \varphi(y)$$

for each x, y in G. Hence and from (10), we have

$$\int_K (g(xl(y)) - \varphi(xl(y)))dm_K(l) = (g(x) - \varphi(x))\omega(y),$$

that is, $g = \varphi + \psi$, where $\psi : G \to \mathbb{C}$ satisfies (11) for each x, y in G. The theorem is proved. □

We note that the above results can be generalized to some extent. In fact, we haven't used inverses neither in G, nor in K. It follows that similar results can be obtained if we suppose that G and K are just some types of topological semigroups satisfying reasonable conditions so that the existence of an invariant integral on K and – in the case of the general equation (2) – an invariant mean on G is guaranteed.

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