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ON ADDITION THEOREMS

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Presented by J. Aczél, F.R.S.C.

ABSTRACT. In this note polynomially additive functions are defined and characterized.

Heuristically, an addition theorem is a relation between the function values $f(x)$, $f(y)$, $f(x+y)$ of some function f, which shows how to calculate the value of f at $x+y$ knowing its values at x and y. More generally, we may call a relation an addition theorem, if it expresses the value of a function f at $x+y$ using the values of some other functions $g,h,\ldots$ at x and y. For instance, addition theorems of this type for the trigonometric functions are standard subjects of undergraduate courses in mathematics. On the other hand, these types of addition theorems can be considered as functional equations if the question is : which functions have a given addition theorem ? From this point of view the simplest addition theorems are expressed by the classical Cauchy-equations

$$a(x+y) = a(x) + a(y) \tag{1}$$

and

$$m(x+y) = m(x)m(y) \quad . \tag{2}$$

Solutions of (1) are called additive, and those of (2) are called exponential functions.

A common property of the above mentioned addition theorems is that for the computation of the function value at $x+y$ one uses only two operations : addition and multiplication, that is, $f(x+y)$ is a polynomial of the values of some fixed functions at x and at y. There are other classical and "nice" addition theorems lacking this property : for instance, addition theorems for tangent and cotangent, and for some elliptic functions. It is natural to pose the problem : determine all functions which possess a "polynomial addition theorem" in the above mentioned sense.

First we need a definition. $\mathbb{C}$ denotes the set of complex numbers.

Definition. Let G be an abelian group and $f: G \to \mathbb{C}$ a function. We say that f is polynomially additive, if there exist nonnegative integers n, m, a complex polynomial P in $n+m$ variables, and functions $g_1,\ldots,g_n,h_1,\ldots,h_m : G \to \mathbb{C}$ such that

$$f(x+y) = P(g_1(x),\ldots,g_n(x),h_1(y),\ldots,h_m(y)) \tag{3}$$

holds for all x,y in G. A functional equation of the form (3) is called a polynomial addition theorem.

Now our problem is the following : given G, determine all polynomially additive functions. Trivially, all additive and exponential functions are polynomially additive. But we can go one step further : let k,l be nonnegative integers, Q a complex polynomial in $k+l$ variables and $a_1,\ldots,a_k,m_1,\ldots,m_l : G \to \mathbb{C}$ functions, where a_i is additive and m_j is exponential ($i = 1,\ldots,k$; $j = 1,\ldots,l$). Then the function $f: G \to \mathbb{C}$ defined by

$$f(x) = Q(a_1(x),\ldots,a_k(x),m_1(x),\ldots,m_l(x)) \tag{4}$$

is polynomially additive.

The proof is trivial by easy calculation. Functions of the form (4) are called exponential polynomials. Hence we have the following.

THEOREM 1. On an arbitrary abelian group any complex valued exponential polynomial is polynomially additive.

The complete solution of our above mentioned problem is that the converse statement is also true.

THEOREM 2. On an arbitrary abelian group any complex valued polynomially additive function is an exponential polynomial.

Proof. Suppose, that $f: G \to \mathbb{C}$ is polynomially additive, that is (3) holds for all x,y in G. This means, that f satisfies a functional equation of the form

$$f(x+y) = \sum_{i=1}^{N} G_i(x)H_i(y)$$

with some functions $G_i, H_i : G \to \mathbb{C}$ ($i = 1,\ldots,N$). Hence, f belongs to a finite dimensional translation invariant subspace of all complex valued functions on G ; it follows from the results of [2] (see also [1] , [3]) that f is an exponential polynomial.

We remark finally that the notion of "polynomially additive function" can be modified by replacing P in (3) by elements of other function classes. For instance, in the real case it would be interesting to determine all "rationally additive functions", where P in (3) is a rational function. In the complex case the same can be studied concerning entire functions. In the general case, the problem can be modified by asking for the solutions of (3) where P is a generalized polynomial.

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