ON A WAVE-TYPE FUNCTIONAL EQUATION

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Abstract. In this work the functional equation

$$f(x+t,y+t)+f(x,y)=f(x+t,y)+f(x,y+t)$$

is solved, where x, y vary in an abelian group G and t varies in a subset T of G, which is a uniqueness set for exponential polynomials in the sense, that if an exponential polynomial vanishes on T, then it vanishes identically.

In this work we deal with the functional equation

$$f(x+t,y+t)+f(x,y)=f(x+t,y)+f(x,y+t), \tag{1}$$

which is a difference-analogue of the wave-equation. Equation (1) and equivalents of it have been treated by several authors ([1], [2], [3], [4], [6]) and in [6] the general solution of (1) on an abelian group G is given in the form

$$f(x,y)=\varphi(x)+\psi(y)+B(x,y) \tag{2}$$

for all x, y in G, where $B : G \times G \to \mathbb{C}$ is biadditive and skew-symmetric, and $\varphi, \psi : G \to \mathbb{C}$ are arbitrary functions.

Here we prove the same for a local version of (1), that is, we do not suppose (1) to hold for all x, y in G, but for all x in G and y in T, where T is a possibly smaller set. Our result is based on the method indicated in [11], [12].

If G is an abelian group, then the homomorphisms of G into the additive group of the complex numbers, or into the multiplicative group of nonzero complex numbers are called *additive* or *exponential functions*, respectively. The elements of the complex algebra generated by the additive functions, or by the additive and exponential functions, are called *polynomials* or *exponential polynomials*, respectively. Hence the general form of an exponential polynomial is

$$x \mapsto \sum_{i=1}^{n} P_i(a_1(x), a_2(x), \dots, a_k(x))m_i(x),$$

where P_i is a complex polynomial in k variables, further $a_j : G \to \mathbb{C}$ is additive and $m_i : G \to \mathbb{C}$ is exponential $(i = 1, 2, \dots, n; j = 1, 2, \dots, k)$. If the sum contains only one nonzero term, say $P_i = P$, and P is a monomial, then the respective exponential

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polynomial is called an *exponential monomial*, and if, in addition, $m_i = 1$, then we call it a *monomial.* Hence any exponential monomial has the form

$$x \mapsto a_1(x)^{\alpha_1} a_2(x)^{\alpha_2} \dots a_k(x)^{\alpha_k} m(x),$$

where a_i is additive and m is an exponential, and this is a monomial if and only if $m = 1$.

The basic theorem on spectral synthesis for finitely generated discrete abelian groups is the following.

Theorem 1. *If a translation invariant linear space of complex valued functions on a finitely generated discrete abelian group is given, which is closed with respect to the pointwise convergence, then any element of this space is the pointwise limit of a net of linear combinations of exponential monomials, which belong to this space. (*[5]*)*

Let G be an abelian group. We say that a subset $T \subseteq G$ is a *uniqueness set*, if for any exponential polynomial $f : G \to \mathbb{C}$, $f = 0$ on T implies $f = 0$ on G. Obviously G is a uniqueness set, and by the results of [9], [10], if G is a topological group which is generated by every neighborhood of zero (for instance, it is connected), then any nonempty open set is a uniqueness set, or if G is a locally compact abelian group which is generated by every neighborhood of zero, then any measurable set of positive measure is a uniqueness set.

In order to prove our main theorem we need a lemma, which can be generalized, but here the following simple version is also satisfactory.

Lemma 2. *Let G be a commutative semigroup, and let $a, b, c : G \to \mathbb{C}$ additive functions. Then*
i) if $ab = 0$, then $a = 0$ or $b = 0$;
ii) if $abc = 0$, then $a = 0$ or $b = 0$ or $c = 0$.

Proof.
i) If $ab = 0$, then

$$0 = a(x+y)b(x+y) = a(x)b(y) + a(y)b(x)$$

for all x, y in G, hence if $a \neq 0$, then $b = \lambda a$ with some complex λ, and by $ab = 0$ it follows $\lambda = 0$ and $b = 0$.

ii) If $abc = 0$, then

$$0 = a(x+y)b(x+y)c(x+y) = a(x)b(x)c(y) + a(y)b(y)c(x) + a(x)b(y)c(x) +$$

$$+ a(y)b(x)c(y) + a(y)b(x)c(x) + a(x)b(y)c(y)$$

for all x, y in G. If a polynomial is zero, then the sum of all its terms of the same degree must be zero. Hence

$$a(x)b(x)c(y) + a(x)c(x)b(y) + b(x)c(x)a(y) = 0$$

for all x, y in G. By (i) we may suppose that a, b, c are linearly dependent, for instance $c = \lambda a + \mu b$. Then substitution into the last equation yields

$$[2\lambda a(x)b(x) + \mu b(x)^2]a(y) + [2\mu a(x)b(x) + \lambda a(x)^2]b(y) = 0$$

for all x, y in G. If a and b are linearly dependent, then $b = \delta a$, and substitution into $abc = 0$ gives $a = 0$. If a and b are linearly independent, then

$$2\lambda a(x)b(x) + \mu b(x)^2 = 0$$

and

$$2\mu a(x)b(x) + \lambda a(x)^2 = 0,$$

which implies $\mu^2 b(x)^2 = \lambda^2 a(x)^2$ for all x, y in G, and the substitution $x \to x+y$ gives that a and b are linearly dependent, a contradiction. Hence our lemma is proved.

We note that the above lemma also follows from the results of [7], [8].

Our main theorem follows.

Theorem 3. *Let G be an abelian group, $T \subseteq G$ a uniqueness set and $f : G\times G \in \mathbb{C}$ a function satisfying* (1) *for all x, y in G and for all t in T. Then f has the form* (2) *for all x, y in G, where $B : G \times G \to \mathbb{C}$ is biadditive and skew-symmetric, and $\varphi, \psi : G \to \mathbb{C}$ are arbitrary functions.*

Proof. First we suppose that G is finitely generated. It is obvious, that all solutions $f : G \times G \to \mathbb{C}$ of (1) form a translation invariant linear space, which is closed with respect to pointwise convergence. On the other hand, as the limit of any pointwise convergent net of functions of the form (2) has the form (2) again, by Theorem 1. it is enough to show, that all exponential monomials, which are solutions of (1) have the form (2). First we show this for exponential solutions. Any exponential on $G \times G$ has the form $m(x, y) = m_1(x)m_2(y)$, where m_1 and m_2 are exponentials on G. Substitution into (1) gives, that the exponential polynomial

$$t \mapsto m_1(t)m_2(t) + 1 - m_1(t) - m_2(t)$$

vanishes on T, hence, by assumption, on G. As different exponentials are linearly independent (see [11], [12]), it follows that either $m_1 = 1$ or $m_2 = 1$, hence m is of the form (2) with $B = 0$.

Suppose now, that $(x, y) \mapsto p(x, y)m(x)$ is a solution of (1) with some monomial p. Then substitution into (1) gives that for any fixed x, y in G the exponential polynomial

$$t \mapsto [p(x + t, y + t) - p(x + t, y)]m(t) + [p(x, y) - p(x, y + t)]$$

vanishes on T, hence, by assumption, on G. But different exponentials are linearly independent over the ring of polynomials (see [11], [12]) , hence $m \neq 1$ implies that p depends only on the first variable. This means, that the corresponding exponential monomial depends only on the first variable, too. Similarly, if we take an exponential monomial solution of (1) of the form $(x, y) \mapsto p(x, y)m(y)$ with $m \neq 1$, then it follows, that p, and also the exponential monomial depends only on the second variable. Hence all exponential monomials in the solution space of (1), corresponding to exponentials different from 1 have the form (2) with $B = 0$. What is left, is to deal with the case $m = 1$, that is, to show that all monomial solutions p of (1) have the form (2). If p is of degree one, then it has the form

$$p(x, y) = a(x) + b(y)$$

with some additive functions $a, b : G \to \mathbb{C}$ and this is the desired form.

If p is of degree two, then it has the form

$$p(x, y) = [a(x) + b(y)][c(x) + d(y)]$$

with some additive functions $a, b, c, d : G \to \mathbb{C}$, and substitution into (1) gives, that necessarily

$$a(t)d(t) + b(t)c(t) = 0$$

for all t in T, hence, by assumption, in G. This means that

$$p(x, y) = a(x)c(x) + b(y)d(y) + [a(x)d(y) + b(y)c(x)],$$

where the function in the square brackets is biadditive and skew-symmetric. Hence p has the form (2).

Now we consider monomial solutions of degree at least three, which have the form

$$p(x, y) = [a(x) + b(y)][c(x) + d(y)][e(x) + f(y)]q(x, y)$$

with some additive functions $a, b, c, d, e, f : G \to \mathbb{C}$, and with some nonzero monomial $q : G \times G \to \mathbb{C}$. We suppose, that $p \neq 0$. We show, that if p is a solution of (1), then it has the form $p(x, y) = \varphi(x)$ or $p(x, y) = \psi(y)$, depending on if $a \neq 0$ or $a = 0$. Suppose, that $a \neq 0$. Obviously we may assume, that $q = 1$. Substitution into (1) gives that

$$a(t)c(t)f(t) + b(t)d(t)e(t) = 0 \tag{3}$$

and from the case of monomials of degree two we know

$$a(t)d(t) + b(t)c(t) = 0 \tag{4}$$

$$a(t)f(t) + b(t)e(t) = 0 \tag{5}$$

$$c(t)f(t) + d(t)e(t) = 0 \tag{6}$$

for all t in T, and hence by assumption, for all t in G. From (6) and (3) it follows $(a(t) - b(t))c(t)f(t) = 0$. From the Lemma, if $c \neq 0$, $f \neq 0$, then $a = b$. Then (4) implies $d = -c$ and (5) implies $f = -e$, and from (3) we arrive at a contradiction. Hence $c = 0$ or $f = 0$. If $c = 0$, then by (6) $e = 0$, and by (5) $a = 0$, a contradiction. Hence, if $a \neq 0$, then $f = 0$, and by (5) $b = 0$, and by (6) $d = 0$, hence our theorem is proved in the case when G if finitely generated.

For the general case, let F be any finitely generated subgroup of G. Then the restriction of f to $F \times F$ has the form (2), that is

$$f|_F(x, y) = \varphi_F(x) + \psi_F(y) + B_F(x, y)$$

holds for any x, y in F, where $\varphi_F, \psi_F : F \to \mathbb{C}$ are arbitrary functions and $B_F : F \times F \to \mathbb{C}$ is a biadditive, skew-symmetric function. It is clear, that $\varphi_F(x) = f(x, 0)$ e's $\psi_F(y) = f(0, y)$ holds for any x, y in F. Further the relation

$$B_F(x, y) = f|_F(x, y) - f(x, 0) - f(0, y),$$

which is valid for any x, y in F shows, that if $F_1 \subseteq F_2$ are finitely generated subgroups of G, then $B_{F_1}(x, y) = B_{F_2}(x, y)$ holds for all x, y in F_1. This implies that by defining

$$B(x, y) = B_F(x, y)$$

for any x, y in F we obtain a biadditive, skeq-symmetric function on $G \times G$ for which

$$f(x, y) = \varphi(x) + \psi(y) + B(x, y)$$

holds for any x, y in G and the theorem is proved.

Corollary 4. *Let G be an abelian group and $f : G \times G \to \mathbb{C}$ a function satisfying* (1) *for all x, y, t in G. Then f has the form* (2).

Corollary 5. *Let G be an abelian topological group which is generated by every neighborhood of zero. If $f : G \times G \to \mathbb{C}$ is a function satisfying* (1) *for all x, y in G and for all t in a nonempty open subset of G, then f has the form* (2).

Corollary 6. *Let G be a locally compact abelian group which is generated by every neighborhood of zero. If $f : G \times G \to \mathbb{C}$ is a function satisfying* (1) *for all x, y in G and for all t in a measurable subset of G of positive measure, then f has the form* (2).

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