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L. Székelyhidi

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INSTITUTUM MATHEMATICUM UNIVERSITATIS DEBRECENIENSIS
HUNGARIA

On a class of a linear functional equations

By LÁSZLÓ SZÉKELYHIDI (Debrecen)

1. Introduction. In this paper we deal with a class of linear functional equations on some types of Abelian groups. Our aim is firstly, to find the general solution of these equations, and secondly, to find the solutions satisfying some regularity conditions if the groups in question are topological.

In the first part we give the general solution of some functional equations by means of polynomials on groups. The notion of polynomial on a group has been introduced by S. MAZUR, W. ORLICZ [8], M. FRÉCHET [3] and G. VAN DER LIJN [11]. The role played by these functions is similar to that played by polynomials on the real line. In particular we determine the general form of polynomials on groups by means of multiadditive functions generalizing the results of M. A. MCKIERNAN [9].

In the second part we determine the regular solutions in case of topological groups by showing that the usual regularity conditions imply continuity. In particular we extend some well-known theorems ([6], [7], [10]). The interest of the method used in this work is that the statements about regularity are derived from the explicit general solution rather than from the equation.

Equations of similar type have been dealt with in [4] and [6] too.

2. Notations and terminology. Throughout this paper **R** will denote the set of reals, **C** the set of complex numbers. If G is a group, the group operation will be denoted by "+" even if G is not commutative. A group is said to be torsion-free if it does not contain any element of finite order except zero. If x is an element of a group and n is a positive integer, then nx denotes the sum $x+x+\ldots+x$ (n times), and if n is a negative integer, then $nx=-(-n)x$. A group is said to be divisible, if for every element x of it there exists an element y with $ny=x$. If a commutative group is divisible and torsion-free then it is a linear space over the rationals.

If G, S are (not necessarily commutative) groups and $f\colon G\to S$ is a function, then for every $y\in G$ let

$$T_y f(x) = f(x+y) \quad (x\in G),$$

and for every $y_1, y_2, \ldots, y_n \in G$ let

$$\Delta^n_{(y_1,\ldots,y_n)} = (T_{y_n}-T_0)(T_{y_{n-1}}-T_0)\ldots(T_{y_1}-T_0).$$

In particular for $y\in G$

$$\Delta^n_y = (T_y - T_0)^n.$$

A function $f\colon G\to S$ is said to be a polynomial of degree at most n, if

$$\Delta^{n+1}_y f(x) = 0$$

for every $x, y \in G$. It is said to be a monomial of degree n, if

$$\Delta_y^n f(x) = n! f(y)$$

for every $x, y \in G$. It is obvious that every monomial of degree at most n is a polynomial of degree at most n.

If G, S are groups and n is a positive integer, then a function $A: G^n \to S$ is said to be n-additive, if it is a homomorphism in each variable. It is said to be symmetric, if it takes the same value at every permutation of its variables. Let $A: G^n \to S$ be a function, then the function φ defined by

$$\varphi(x) = A(x, x, \ldots, x) \quad (x \in G)$$

is said to be the diagonal of A, and is denoted by $D(A)$. Further let

$$A_k(x, y) = A(\underbrace{x}_{1}, \underbrace{x}_{2}, \ldots, \underbrace{x}_{k}, y, y, \ldots, y) \quad (x, y \in G).$$

We use the phrase "0-additive function" for constant functions.

If f is a function, then $\mathrm{Rg} f$ denotes the range of f.

3. Algebraical results

Theorem 3.1. *Let G, S be Abelian groups and let S be torsion-free. Let n be a positive integer and $v_n = n!(n-1)!\ldots 2!$ If $f: G \to S$ is an arbitrary polynomial of degree n, then $v_n \cdot f$ can be written as the sum of monomials of degree at most n.*

The proof can be found in [11].

Theorem 3.2. *Let G, S be Abelian groups and let n be a positive integer. If $\varphi: G \to S$ is a monomial of degree n, then there exists an n-additive symmetric function $A: G^n \to S$ such that $D(A) = n!\varphi$. Further $D(A)$ is a monomial of degree n for an arbitrary n-additive symmetric function A.*

Proof. Let for $y_1, \ldots, y_n \in G$

$$A(y_1, \ldots, y_n) = \Delta^n_{(y_1, \ldots, y_n)} \varphi(0).$$

Then A is symmetric and for $y_1, \bar{y}_1, y_2, \ldots, y_n \in G$ we get

$$A(y_1 + \bar{y}_1, y_2, \ldots, y_n) - A(y_1, y_2, \ldots, y_n) - A(\bar{y}_1, y_2, \ldots, y_n) =$$

$$= \left(\Delta^n_{(y_1+\bar{y}_1, \ldots, y_n)} - \Delta^n_{(y_1, \ldots, y_n)} - \Delta^n_{(\bar{y}_1, \ldots, y_n)}\right)\varphi(0) =$$

$$= (T_{y_1+\bar{y}_1} - T_{y_1} - T_{\bar{y}_1} + T_0) \prod_{i=2}^{n} (T_{y_i} - T_0)\varphi(0) =$$

$$= (T_{y_1} - T_0)(T_{\bar{y}_1} - T_0) \prod_{i=2}^{n} (T_{y_i} - T_0)\varphi(0) = \Delta^{n+1}_{y_1, \bar{y}_1, \ldots, y_n} \varphi(0) = 0$$

so A is n-additive. Further

$$D(A)(y) = \Delta_y^n \varphi(0) = n!\varphi(y) \quad (y\in G).$$

The other statement is trivial.

Corollary 3.3. *Let G, S be Abelian groups and let S be torsion-free. Let n be a positive integer and ν_n as in Theorem 3.1. If $f\colon G\to S$ is a polynomial of degree at most n then there exist $A_k\colon G^k\to S$ k-additive symmetric functions $(k=0, 1, \dots, n)$ such that*

$$\nu_n^2\cdot f = \sum_{k=0}^{n} D(A_k).$$

This representation is unique.

PROOF. We have to prove the uniqueness only. For the proof let $\sum_{k=0}^{n} D(A_k)=0$, where A_k is a k-additive symmetric function $(k=0, 1, \dots, n)$ then for every $x, y_1, \dots, y_n\in G$ we have

$$0 = \Delta^n_{(y_1,\dots,y_n)}\left(\sum_{k=0}^{n} D(A_k)\right)(x) = \Delta^n_{(y_1\dots y_n)} D(A_n)(x) = n!A_n(y_1,\dots,y_n)$$

and so $A_n(y, \dots, y_n)=0$. Similarly we get $A_k=0$ for $k=0, 1, \dots, n-1$.

Lemma 3.4. *Let G, S be Abelian groups, let n be a positive integer, let $A_k\colon G^k\to S$ be k-additive symmetric functions $(k=0, 1, \dots, n)$ and let $f\colon G\to S$ be a monomial of degree n. Then*

i) $\sum_{k=0}^{n} D(A_k)=0$ *implies* $A_k=0$ $(k=0, 1, \dots, n)$

ii) $f\circ\varphi$ *is a monomial of degree n for every homomorphism* $\varphi\colon G\to G$.

PROOF. For the proof of i) see the previous theorem and the proof of the other statement is an easy calculation.

Definition 3.5. Let G, S be Abelian groups, let n be a nonnegative integer. The function $f\colon G\to S$ is said to be of degree n, if there exist functions $f_i\colon G\to S$ and homomorphisms $\varphi_i, \psi_i\colon G\to G$ such that $\operatorname{Rg}\varphi_i\subset\operatorname{Rg}\psi_i$ $(i=1, 2, \dots, n+1)$ and the equation

$$f(x)+\sum_{i=1}^{n+1} f_i(\varphi_i(x)+\psi_i(y)) = 0 \quad (x, y\in G) \tag{1}$$

holds.

Theorem 3.6. *Let G, S be Abelian groups and suppose that G is divisible. Let n be a nonnegative integer. The function $f\colon G\to S$ is of degree n if and only if it is a polynomial of degree n.*

PROOF. $\psi_i(x)=ix$ $(i=1, 2, \dots, n+1)$ is a homomorphism of G and $\operatorname{Rg}\psi_i=G$, hence every polynomial of degree n is a function of degree n. For the converse we have to prove only the following statement: if the function f is of degree n then $\Delta_t f$ is of degree $n-1$ for every $t\in G$, this implies that $\Delta^{n+1}_{(t_1,\dots,t_{n+1})} f$ is identically zero, i.e. f is a polynomial of degree n. Let $t\in G$ be an arbitrary element and $s\in G$ such that $\varphi_{n+1}(t)+\psi_{n+1}(s)=0$. Substituting in (1) $x+t$ for x and $y+s$ for y, and subtracting (1) from the new equation we obtain

$$\Delta_t f(x)+\sum_{i=1}^{n}\Delta_{\varphi_i(t)+\psi_i(s)} f_i[\varphi_i(x)+\psi_i(y)] = 0, \tag{2}$$

this shows that $\Delta_t f$ is of degree $n-1$.

Theorem 3.7. *Let G, S be Abelian groups such that S is torsion-free and let n be a nonnegative integer. Let $\varphi_i\psi_i\colon G\to G$ be homomorphisms $(i=1, 2, \ldots, n+2)$ and let $A_k^{(i)}\colon G^k\to S$ be k-additive symmetric functions $(k=0, 1, \ldots, n, i=1, 2, \ldots, n+2)$. The functions*

$$f_i=\sum_{k=0}^{n} D(A_k^{(i)}) \quad (i = 1, 2, \ldots, n+2)$$

satisfy the functional equation

$$\sum_{i=1}^{n+2} f_i[\varphi_i(x)+\psi_i(y)] = 0 \quad (x, y\in G) \tag{3}$$

if and only if the functions $A_k^{(i)}$ satisfy the relations

$$\sum_{i=1}^{n+2} A_{k,j}^{(i)}(\varphi_i(x), \psi_i(y)) = 0 \quad (x, y\in G) \tag{4}$$

for $j=0, 1, \ldots, n$ and $k=j, j+1, \ldots, n$.

Proof. Let the given functions satisfy equation (3), then using the obvious relation

$$D(A)(x+y) = \sum_{j=0}^{k}\binom{k}{j} A_j(x, y) \quad (x, y\in G)$$

which holds for every k-additive symmetric function A, we get

$$\sum_{i=1}^{n+2}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j} A_{k,j}^{(i)}(\varphi_i(x), \psi_i(y)) = 0 \quad (x, y\in G). \tag{5}$$

Observe that the function $x\to A_{k,j}^{(i)}(\varphi_i(x), \psi_i(y))$ is a monomial of degree j for every $y\in G$, and so (5) and theorem 3.3 imply

$$\sum_{i=1}^{n+2}\sum_{k=j}^{n}\binom{k}{j} A_{k,j}^{(i)}(\varphi_i(x), \psi_i(y)) = 0 \quad (j = 0, \ldots, n, x, y\in G).$$

Similarly we get

$$\sum_{i=1}^{n+2} A_{k,j}^{(i)}(\varphi_i(x), \psi_i(y)) = 0 \quad (x, y\in G) \tag{6}$$

for $j=0, \ldots, n$, $k=j, \ldots, n$ because S is torsion-free. To prove the converse multiply (6) by $\binom{k}{j}$ and add the equations first for $k=j, j+1, \ldots, n$, then for $j=0, 1, \ldots, n$.

Corollary 3.8. *Let G, S be linear spaces over the rationals, let n be a positive integer, let p_i, q_i $(i=1, 2, \dots, n+2)$ be rational numbers and let $A_k^{(i)}$: $G^k \to S$ $(k=0, \dots, n;\ i=1, 2, \dots, n+2)$ be k-additive symmetric functions. The functions*

$$f_i = \sum_{k=0}^{n} D(A_k^{(i)}) \quad i = 1, 2, \dots, n+2$$

satisfy the equation

$$\sum_{i=1}^{n+2} f_i(p_i x + q_i y) = 0 \quad (x, y \in G) \tag{7}$$

if and only if the functions $A_k^{(i)}$ satisfy the relations

$$\sum_{i=1}^{n+2} p_i^j q_i^{k-j} D(A_k^{(i)})(x) = 0 \quad (x \in G) \tag{8}$$

for $j=0, 1, \dots, n$ *and* $k=j, j+1, \dots, n$. (Here $0^0=1$.)

Proof. If G, S are linear spaces over the rationals, then

$$A_j(px, qy) = p^j q^{k-j} A_j(x, y)$$

holds for every k-additive symmetric function A and rationals p, q. By the previous theorem we have to show only that (8) implies the equations

$$\sum_{i=1}^{n+2} A_{k,j}^{(i)}(p_i x, q_i y) = 0 \quad (x, y \in G)$$

for $j=0, 1, \dots, n$; $k=j, j+1, \dots, n$. Let $0 \leqq j \leqq n$ and $j \leqq k \leqq n$ be arbitrary. Then by (8) we get

$$0 = \sum_{i=1}^{n+2} p_i^j q_i^{k-j} D(A_k^{(i)})(x+y) = \sum_{s=0}^{k} \binom{k}{s} \sum_{i=1}^{n+2} p_i^j q_i^{k-j} A_{k,s}^{(i)}(x, y).$$

Hence for $s=0, 1, \dots, k$

$$\sum_{i=1}^{n+2} p_i^j q_i^{k-j} A_{k,s}^{(i)}(x, y) = 0 \quad (x, y \in G).$$

In particular, for $s=j$ we obtain

$$\sum_{i=1}^{n+2} A_{k,j}^{(i)}(p_i x, q_i y) = 0 \quad (x, y \in G).$$

Theorem 3.9. *Let G, S be Abelian groups and suppose that G is divisible and S is torsion-free. Let n be a nonnegative integer and let φ_i, ψ_i: $G \to G$ be homomorphisms of G onto itself such that* $\operatorname{Rg}(\psi_j \circ \psi_i^{-1} - \varphi_j \circ \varphi_i^{-1}) = G$ *for* $i \neq j$ $(i, j=1, 2, \dots, n+1)$. *The functions f_i: $G \to S$ $(i=0, \dots, n+1)$ satisfy the functional equation*

$$f_0(x) + \sum_{i=1}^{n+1} f_i[\varphi_i(x) + \psi_i(y)] = 0 \quad (x, y \in G) \tag{9}$$

if and only if there exist $A_k^{(i)}\colon G^k \to S$ *k-additive symmetric functions*

$$(k = 0, 1, \dots, n;\ i = 0, 1, \dots, n+1)$$

such that

$$f_i = \sum_{k=0}^{n} D(A_k^{(i)}) \quad (i = 0, 1, \dots, n+1) \tag{10}$$

and the equations

$$A_{k,j}^{(0)}(x, 0) + \sum_{i=1}^{n+1} A_{k,j}^{(i)}(\varphi_i(x), \psi_i(y)) = 0 \quad (x, y \in G) \tag{11}$$

hold for $j=0, 1, \dots, n$, $k=j, j+1, \dots, n$.

Proof. Let the functions f_i satisfy (9) and let $0 \leqq i \leqq n+1$ be arbitrary. Further let $u, y \in G$ be arbitrary and $x = \varphi_i^{-1}(u) - \varphi_i^{-1} \circ \psi_i(y)$.

Then by (9) we have

$$0 = f_i(u) + f_0(\varphi_i^{-1}(u) - \varphi_i^{-1} \circ \psi_i(y)) + \sum_{\substack{j=1 \\ j \neq i}}^{n+1} f_j[\varphi_j \circ \varphi_i^{-1}(u) + (\psi_j - \varphi_j \circ \varphi_i^{-1} \circ \psi_i)(y)].$$

Here $\mathrm{Rg}\, \varphi_i^{-1} = G = \mathrm{Rg}\, \varphi_i^{-1} \circ \psi_i$ and

$$\mathrm{Rg}\, \varphi_j \circ \varphi_i^{-1} = G = \mathrm{Rg}(\psi_j \circ \psi_j^{-1} - \varphi_j \circ \varphi_i^{-1}) = \mathrm{Rg}\, (\psi_j - \varphi_j \circ \varphi_i^{-1} \circ \psi_j)$$

so by 3.3 and 3.6 every f_i has the form (10). Conversely by (11) we get

$$\sum_{i=0}^{n+1} A_{k,j}^{(i)}(\varphi_i(x), \psi_i(y)) = 0 \quad (j = 0, \dots, n,\ k = j, \dots, n,\ x, y \in G)$$

and by 3.7 the proof is complete. (Here $\varphi_0(x) = x$, $\psi_0(x) = 0$ for $x \in G$.)

Corollary 3.10. *Let G, S be linear spaces over the rationals. Let n be a non-negative integer and let p_i, q_i be rationals different from zero such that $p_i q_j \neq p_j q_i$ for $i \neq j$ $(i, j = 1, 2, \dots, n+1)$. Then the general solution of the functional equation*

$$f_0(x) + \sum_{i=1}^{n+1} f_i(p_i x + q_i y) = 0 \quad (x, y \in G) \tag{12}$$

has the form

$$f_i(x) = \sum_{k}^{n} D(A_k^{(i)}) \quad (i = 0, 1, \dots, n+1)$$

where the functions $A_k^{(i)}\colon G^k \to S$ are k-additive symmetric ones $(k = 0, 1, \dots, n$, $i = 0, 1, \dots, n+1)$ for which the relations

$$D(A_K^{(0)})(x) + \sum_{i=1}^{n+1} p_i^k D(A_k^{(i)})(x) = 0 \quad (x \in G,\ j = 1, \dots, n-1,\ k = j+1, \dots, n) \tag{13}$$

$$\sum_{i=1}^{n+1} p_i^j q_i^{k-j} D(A_k^{(i)})(x) = 0 \quad (x \in G,\ j = 1, \dots, n-1,\ k = j+1, \dots, n) \tag{14}$$

$$\sum_{i=1}^{n+1} q_i^k D(A_k^{(i)})(x) = 0 \quad (x \in G,\ k = 0, \dots, n) \tag{15}$$

hold.

PROOF. By 3.9 it is sufficient to show that with the notations

$$\varphi_i(x) = p_i x, \quad \psi_i(x) = q_i x \quad (i = 1, \dots, n+1, \ x \in G)$$

(11) is equivalent with (13), (14) and (15). If we write in (11) $j=k$, then we get (13), and with $j=0$ and arbitrary k we obtain (15) from (11). Now let $1 \leqq j < k \leqq n$ and $x, y \in G$ be arbitrary, then from (14) we have

$$0 = \sum_{i=1}^{n+1} p_i^j q_i^{k-j} D(A_k^{(i)})(x) = \sum_{s=0}^{k} \binom{k}{s} \sum_{i=1}^{n+1} p_i^j q_i^{k-j} A_{k,s}^{(i)}(x, y)$$

and hence for $s=0, 1, \dots, k$

$$\sum_{i=1}^{n+1} p_i^j q_i^{k-j} A_{k,s}^{(i)}(x, y) = 0$$

and in particular for $s=j$ we have

$$\sum_{i=1}^{n+1} A_{k,j}^{(i)}(p_i x + q_i y) = 0 \quad (x, y \in G)$$

which completes the proof.

4. Topological results

Lemma 4.1. *Let G be an arbitrary group, let n be a positive integer and $A: G^n \to \mathbf{R}$ be an n-additive symmetric function. If $k_1, k_2, \dots, k_n$ are positive integers, then there exists a positive integer N such that for every $x_1^{(i)}, x_2^{(i)}, \dots, x_{k_i}^{(i)} \in G$ $(i=1, 2, \dots, n)$ the relation*

$$|A(x_1^{(1)} + \dots + x_{k_1}^{(1)}, \dots, x_1^{(n)} + \dots + x_{k_n}^{(n)})| \leqq N \cdot \max_{1 \leqq i_s \leqq k_s} |A(x_{i_1}^{(1)}, x_{i_2}^{(2)}, \dots, x_{i_n}^{(n)})|$$

holds.

The proof is a simple calculation.

Theorem 4.2. *Let G be a topological group, let n be a positive integer and let A_k: $G^k \to \mathbf{R}$ be k-additive symmetric functions for $k=1, 2, \dots, n$. If $\sum_{k=1}^{n} D(A_k)$ is continuous then A_k is continuous for $k=1, 2, \dots, n$.*

PROOF. Let C_k^n denote the set of all combinations of k-th class of the set $\{1, 2, \dots, n\}$ and for $(i_1, i_2, \dots, i_k) \in C_k^n$ we define the function $p_{(i_1, \dots, i_k)}$: $G^n \to G$ as follows:

$$p_{(i_1, \dots, i_k)}(x_1, \dots, x_n) = x_{i_1} + \dots + x_{i_k} \quad (x_1, \dots, x_n) \in G^n.$$

Observe that for $f = \sum_{k=1}^{n} D(A_k)$ we have

$$A_n = \frac{1}{n!} \sum_{k=1}^{n} \sum_{(i_1, \dots, i_k) \in C_k^n}' (-1)^{n-k} f \circ p_{(i_1, \dots, i_k)} \tag{17}$$

If f is continuous, then so is A_n by (17). Then $f-D(A_n)$ is continuous and by (17) A_{n-1} is continuous too. Continuing similarly we get our statement.

Theorem 4.3. *Let G be a topological group, let n be a positive integer and let A_k: $G^k\to\mathbf{R}$ be k-additive symmetric functions for $k=0, 1, \ldots, n$. Let G_0 denote the component of the identity in G. If $\sum_{k=0}^{n} D(A_k)$ is continuous at the identity then A_k is continuous on G_0^k $(k=0, 1, 2, \ldots, n)$.*

Proof. We know by (17) that A_n is continuous at $(0, 0, \ldots, 0)$. It is sufficient to show that any n-additive symmetric function A: $G^n\to\mathbf{R}$ which is continuous at $(0, 0, \ldots, 0)$, is continuous on G_0^n. First we show that for arbitrary $y_2, y_3, \ldots, y_n\in G_0$ the function $x\to A(x, y_2, \ldots, y_n)$ is continuous on G_0. Let $\varepsilon>0$ be given and let $W_1, W_2, \ldots, W_n$ be neighbourhoods of the identity in G such that for every $(x_1, \ldots, x_n)\in W_1\times\ldots\times W_n$ we have $|A(x_1, \ldots, x_n)|<\varepsilon$. It is known that in G every neighbourhood of the identity generates G_0. Let $z_1^{(i)}, z_2^{(i)}, \ldots, z_{k_i}^{(i)}\in W_i$ such that $z_k^{(i)}+\ldots+z_{k_i}^{(i)}=y_i$ $(i=2, \ldots, n)$ and let N be the same positive integer as in 4.1. Let $U\subset W_1$ be a neighbourhood of the identity in G such that for $x_1, x_2, \ldots, x_N\in U$ we have $x_1+\ldots+x_N\in W_1$. Then $h\in U$ implies $N\cdot h\in W_1$ and we have

$$|A(h, y_2, \ldots, y_n)| = |A(h, z_1^{(2)}+\ldots+z_{k_2}^{(2)}, \ldots, z_1^{(n)}+\ldots+z_{k_n}^{(n)})| \leqq$$
$$\leqq N\cdot \operatorname{Max}|A(h, z_{i_2}^{(2)}, \ldots, z_{i_n}^{(n)})| < \operatorname{Max}|A(Nh, z_{i_2}^{(2)}, \ldots, z_{i_n}^{(n)})| < \varepsilon.$$

We can prove similarly that for every $1<k<n$ the function

$$(x_1, \ldots, x_k) \to A(x_1, \ldots, x_k, y_k, \ldots, y_n)$$

is continuous at $(\underset{1}{\underbrace{0}}, \underset{2}{\underbrace{0}}, \ldots, \underset{k}{\underbrace{0}})$, whenever $y_j\in G_0$ $(j=k+1, \ldots, n)$. Now let $\varepsilon>0$ be arbitrary $(x_1, \ldots, x_n)\in G^n$ and let U be a neighborhood of the identity in G such that if $h\in U$ and $x_i=h$ for some i then $|A(x_1, \ldots, x_n)|<\varepsilon$. If $W=U\times\ldots\times U$, and $(h_1, \ldots, h_n)\in W$, then

$$|A(x_1+h_1, \ldots, x_n+h_n)-A(x_1, \ldots, x_n)| < C\cdot\varepsilon$$

where the constant C depends on n only. This completes the proof.

Theorem 4.4. *Let G be a locally compact group, let n be a positive integer and let A_k: $G^k\to\mathbf{R}$ be k-additive symmetric functions for $k=0, 1, 2, \ldots, n$. Let G_0 denote the component of the identity in G. If $\sum_{k=}^{n} D(A_k)$ is bounded in some neighbourhood of the identity, then A_k is continuous on G_0^k $(k=0, 1, 2, \ldots, n)$.*

Proof. By (17) A_n is bounded in some neighbourhood W of $(0, 0, \ldots, 0)$ that is $(x_1, \ldots, x_n)\in W$ implies

$$|A_n(x_1, \ldots, x_n)| \leqq K.$$

Let U be a neighbourhood of the identity for which $U\times\ldots\times U\subset W$ and suppose that $D(A_n)$ is not continuous at the identity. Then there exists an $\varepsilon>0$ such that every neighbourhood of the identity contains an element z satisfying the inequality

$$|D(A_n)(z)| \geqq \varepsilon.$$

Let $m > \sqrt[n]{\frac{K}{\varepsilon}}$ be a natural number and $V \subset U$ be a neighbourhood of the identity such that $z \in V$ implies $mz \in U$. Choose an element $z \in V$ with the property $|D(A_n)(z)| \geqq \varepsilon$. Then $mz \in U$, $(mz, mz, \ldots, mz) \in W$ and

$$|A_n(mz, \ldots, mz)| = m^n |D(A_n)(z)| > K$$

which is a contradiction. Thus by 4.3 the proof is complete.

Lemma 4.5. *Let G be a locally compact group, $K \subset G$ be a compact set and let λ denote the right Haar-measure. Then the function*

$$(x_1, \ldots, x_n) \to \lambda[(K-x_1) \cap \ldots \cap (K-x_n)]$$

is continuous on G^n for every positive integer n. (See [5], [6].)

Theorem 4.6 *Let G be a locally compact group, let n be a positive integer and let A_k: $G^k \to \mathbf{R}$ be k-additive symmetric functions for $k=0, 1, \ldots, n$. Let G_0 denote the component of the identity in G. If $\sum_{k=0}^{n} D(A_k)$ is bounded or measurable on some measurable subset of G_0 of positive measure, then A_k is continuous on G_0^k $(k=0, 1, \ldots, n)$.*

Proof. The second statement is a consequence of the first one. For if $f = \sum_{k=0}^{n} D(A_k)$ and f is measurable on some measurable set with positive measure, then it is measurable on some compact set of positive measure. By Lusin's theorem (see e.g. [2]) the restriction of f on some compact set of positive measure is continuous, and so it is bounded on this compact set. Now let f be bounded on some measurable set of positive measure, then it is bounded on some compact set K, with $\lambda K > 0$. The function $x \to \lambda[K \cap (K-x) \cap \ldots \cap (K-nx)]$ is continuous on G by lemma 4.5, and its value at the identity is positive. Thus we can choose a neighbourhood U of the identity such that $\lambda[K \cap (K-x) \cap \ldots \cap (K-nx)] > 0$ for $x \in U$. Then there exists a $y \in K \cap (K-x) \cap \ldots \cap (K-nx)$ i.e. $y \in K, y \in K-x, \ldots, y+nx \in K$. Observe that

$$D(A_n)(x) = \frac{1}{n!} \Delta_x^n f(y) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(y+kx)$$

which implies the boundedness of $D(A_n)$ on U. Then by 4.4 A_n is continuous on G_0^n and by repeating this argument for $\sum_{k=0}^{n-1} D(A_k)$ instead of $\sum_{k=0}^{n} D(A_k)$ the proof is complete.

It is easy to see that the results of this paragraph remain valid if the ranges of the functions in question are in some linear topological space.

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