Functional equations on the $SU(2)$-hypergroup

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Abstract

We consider classical functional equations on a special hypergroup which is related to continuous unitary irreducible representations of the special linear group in two dimensions.

1 Introduction

Functional equations on hypergroups have been treated in [6], [7]. In this paper we study functional equations on a special hypergroup, which is related to the set of continuous unitary irreducible representations of the group $G = SU(2)$, the *special linear group* in two dimensions. We show how to determine all exponentials, additive

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functions and generalized moment function sequences on this hypergroup. Moment functions on other types of hypergroups have been described in [3], [4] and [5]. The definition of the underlying hypergroup is taken from [1].

If G is a compact topological group then its dual object $\widehat{G}$ consists of equivalence classes of continuous irreducible representations of G. For any two classes U, V of this type their tensor product can be decomposed into its irreducible components $U_1, U_2, \ldots, U_n$ with the respective multiplicities $m_1, m_2, \ldots, m_n$ (see [2]). We define convolution on $\widehat{G}$ by

$$\delta_U * \delta_V = \sum_{i=1}^{n} \frac{m_i\, d(U_i)}{d(U)\, d(V)}\, \delta_{U_i} \tag{1}$$

where $d(U)$ denotes the dimension of U and δ_U is the Dirac measure concentrated at U. Then $\widehat{G}$ with this convolution and with the discrete topology is a commutative hypergroup.

In the special case of $G = SU(2)$ the dual object $\widehat{G}$ can be identified with the set $\mathbb{N}$ of natural numbers as it is indicated in [1]: the set of equivalence classes of continuous unitary irreducible representations of $SU(2)$ is given by $\{T^{(0)}, T^{(1)}, T^{(2)}, \ldots\}$, where $T^{(n)}$ has dimension $n+1$, and we identify this set with $\mathbb{N}$.

For every m, n in $\mathbb{N}$ the tensor product of $T^{(m)}$ and $T^{(n)}$ is unitary equivalent to

$$T^{(|m-n|)} \bigoplus T^{(|m-n|+2)} \bigoplus \cdots \bigoplus T^{(m+n)} \,. \tag{2}$$

The convolution is given by

$$\delta_m * \delta_n = \sum_{k=|m-n|}^{m+n}{}' \, \frac{k+1}{(m+1)(n+1)} \delta_k \,, \tag{3}$$

where the prime denotes that every second term appears in the sum, only. With this convolution $\mathbb{N}$ becomes a discrete commutative hypergroup, and since all the $T^{(n)}$

are self-conjugate, the hypergroup is in fact Hermitian. We call this hypergroup the *$SU(2)$-hypergroup.*

2 Exponential functions on the $SU(2)$-hypergroup

In this section we describe the exponential functions on the $SU(2)$-hypergroup. We recall that the function $M : \mathbb{N} \to \mathbb{C}$ is an exponential if and only if it satisfies

$$M(m)M(n) = M(m * n) = \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)} M(k) \tag{4}$$

for all natural numbers m, n.

Theorem 1. *The function $M : \mathbb{N} \to \mathbb{C}$ is an exponential on the $SU(2)$-hypergroup if and only if there exists a complex number λ such that*

$$M(n) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda} \tag{5}$$

holds for each natural number n. (Here $\lambda = 0$ corresponds to the exponential $M = 1$.)

Proof. Let $M : \mathbb{N} \to \mathbb{C}$ be a solution of (4) and let $f(n) = (n+1)M(n)$ for each n in $\mathbb{N}$. Then we have

$$f(m)f(n) = \sum_{k=|m-n|}^{m+n}{}' f(k)$$

for each m, n in $\mathbb{N}$. With $m = 1$ it follows that f satisfies the following second order homogeneous linear difference equation

$$f(n+2) - f(1)f(n+1) + f(n) = 0 \tag{6}$$

for each n in $\mathbb{N}$ with $f(0) = 1$.

Suppose that $f(1) = 2$. Then from (6) we infer that $f(n) = n + 1$ and $M = 1$ which corresponds to the case $\lambda = 0$ in (5). Otherwise $f(1) \neq 2$ and let $\lambda \neq 0$ be a complex number with $f(1)=2\cosh\lambda$. Then we have that

$$f(n)=\alpha e^{n\lambda}+\beta e^{-n\lambda}$$

holds for any n in $\mathbb{N}$ with some complex numbers α,β satisfying $\alpha+\beta=1$. It is easy to see that in this case

$$f(n)=\frac{\sinh[(n+1)\lambda]}{\sinh\lambda}$$

holds for each n in $\mathbb{N}$. Finally, we have

$$M(n)=\frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda}\,.$$

Conversely, it is easy to check that any function M of the given form is an exponential on the $SU(2)$-hypergroup, hence the theorem is proved. □

3 Additive functions on the $SU(2)$-hypergroup

Now we describe the additive functions on the $SU(2)$-hypergroup. We recall that the function $A:\mathbb{N}\to\mathbb{C}$ is an additive function if and only if it satisfies

$$A(m)+A(n)=A(m*n)=\sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)}A(k) \tag{7}$$

for all natural numbers m,n.

Theorem 2. *The function $A:\mathbb{N}\to\mathbb{C}$ is an additive function on the $SU(2)$-hypergroup if and only if there exists a complex number c such that*

$$A(n)=\frac{c}{3}n(n+2)$$

holds for each natural number n.

Proof. Let $A:\mathbb{N}\to\mathbb{C}$ be a solution of (7) and let $f(n)=(n+1)A(n)$ for each n in $\mathbb{N}$. Then we have

$$(n+1)f(m)+(m+1)f(n)=\sum_{k=|m-n|}^{m+n}{}' f(k)$$

for each m, n in $\mathbb{N}$. With $m = 1$ it follows that f satisfies the following second order homogeneous linear difference equation

$$f(n+2) - 2f(n+1) + f(n) = 2c(n+2)$$

for each n in $\mathbb{N}$ with $f(0) = 0$ and $f(1) = 2c$. As the second difference of f is linear it follows that f is a cubic polynomial and simple computation gives that A has the desired form.

Conversely, it is easy to check that any function A of the given form is an additive function on the $SU(2)$-hypergroup, hence the theorem is proved. □

4 Generalized moment functions on the $SU(2)$-hypergroup

Finally we describe the generalized moment functions on the $SU(2)$-hypergroup. Let N be a nonnegative integer. We recall that the functions $\varphi_0, \varphi_1, \dots, \varphi_N : \mathbb{N} \to \mathbb{C}$ form a generalized moment function sequence if and only if they satisfy

$$\varphi_k(m * n) = \sum_{j=0}^{k} \binom{k}{j} \varphi_j(m)\varphi_{k-j}(n) \tag{8}$$

for all natural numbers m, n and for $k = 0, 1, \dots, N$.

Making use of the results in Section 2. we introduce the function

$$\Phi(n, \lambda) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda} \tag{9}$$

for each n in $\mathbb{N}$ and $\lambda \neq 0$ in $\mathbb{C}$, while $\Phi(n, 0) = 1$ for each n in $\mathbb{N}$. The function $\Phi : \mathbb{N} \times \mathbb{C} \to \mathbb{C}$ is called an *exponential family* for the $SU(2)$-hypergroup: each exponential on this hypergroup has the form $n \mapsto \Phi(n, \lambda)$ with some unique λ in $\mathbb{C}$, and, conversely, the function $n \mapsto \Phi(n, \lambda)$ is an exponential on the $SU(2)$-hypergroup for every complex λ.

Theorem 3. *Let K denote the $SU(2)$-hypergroup and Φ the exponential family given by (9). The functions $\varphi_0, \varphi_1, ..., \varphi_N : K \to \mathbb{C}$ form a generalized moment sequence of order N on K if and only if there exist complex numbers c_j for $j = 1, 2, \ldots, N$ such that*

$$\varphi_k(n) = \frac{d^k}{dt^k} \Phi(n, f(t))(0)$$

holds for each n in $\mathbb{N}$ and for $k = 0, 1, \ldots, N$, where

$$f(t) = \sum_{j=0}^{N} \frac{c_j}{j!} t^j \tag{10}$$

for each t in $\mathbb{C}$.

Proof. First we note that, by (3), we have for $n \geq 1$

$$\delta_n * \delta_1 = \sum_{k=n-1}^{n+1}{}' \frac{k+1}{2(n+1)} \delta_k = \frac{n}{2(n+1)} \delta_{n-1} + \frac{n+2}{2(n+1)} \delta_{n+1}\,, \tag{11}$$

hence, by 3.2.1 Proposition in [1], K is a polynomial hypergroup, that is, there exists a sequence $(P_n)_{n \in \mathbb{N}}$ of polynomials such that $\deg P_n = n$ for $n = 0, 1, \ldots$, there exists an x_0 in $\mathbb{R}$ such that $P_n(x_0) = 1$ for $n = 0, 1, \ldots$, and

$$P_n(x) P_m(x) = \sum_{k=0}^{\infty} c(m, n, k) P_k(x) \tag{12}$$

holds for each x in $\mathbb{R}$ and m, n in $\mathbb{N}$ with some nonnegative numbers $c(m, n, k)$, further we have

$$\delta_m * \delta_n = \sum_{k=0}^{\infty} c(m, n, k) \delta_k \tag{13}$$

for each m, n in $\mathbb{N}$. Here we shall determine this sequence of polynomials.

Our basic observation is that the function $\lambda \mapsto \Phi(n, \lambda)$ is a polynomial of $\cosh \lambda$ of degree n for each n in $\mathbb{N}$. We apply mathematical induction. For $n = 0$ and $n = 1$

we have by (9)

$$\Phi(0,\lambda) = \frac{\sinh\lambda}{\sinh\lambda} = 1\,,$$
$$\Phi(1,\lambda) = \frac{\sinh(2\lambda)}{2\sinh\lambda} = \cosh\lambda\,.$$

Suppose, that for $k = 0, 1, \ldots, n$ there exists a polynomial P_k of degree k such that

$$\Phi(k,\lambda) = P_k(\cosh\lambda) \tag{14}$$

holds. Clearly $P_0(x) = 1$ and $P_1(x) = x$. Then, by equation (11), we have

$$P_n(\cosh\lambda)\cosh\lambda = \frac{n}{2(n+1)}P_{n-1}(\cosh\lambda) + \frac{n+2}{2(n+1)}\Phi(n+1,\lambda)\,, \tag{15}$$

that is

$$\Phi(n+1,\lambda) = \frac{2(n+1)}{n+2}P_n(\cosh\lambda)\cosh\lambda - \frac{n}{n+2}P_{n-1}(\cosh\lambda)\,, \tag{16}$$

and here the right hand side is a polynomial of degree $n+1$ in $\cosh\lambda$:

$$P_{n+1}(x) = \frac{2(n+1)}{n+2}xP_n(x) - \frac{n}{n+2}P_{n-1}(x)\,,$$

hence

$$\Phi(n+1,\lambda) = P_{n+1}(\cosh\lambda)\,,$$

which was to be proved.

Finally, we have for all m, n in $\mathbb{N}$ and λ in $\mathbb{C}$

$$P_n(\cosh\lambda)P_m(\cosh\lambda) = \Phi(n,\lambda)\Phi(m,\lambda) = \Phi(n*m,\lambda) =$$

$$= \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)}\,\Phi(k,\lambda) = \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)}\,P_k(\cosh\lambda)\,,$$

which implies

$$P_n(x)P_m(x) = \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)}\,P_k(x)$$

for each x in $\mathbb{R}$ and m, n in $\mathbb{N}$. This means that K is the polynomial hypergroup associated to the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. Then, by Theorem 4. in [4], our statement follows. □

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