Functional equations on the $SU(2)$-hypergroup

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**Abstract**

We consider classical functional equations on a special hypergroup which is related to continuous unitary irreducible representations of the special linear group in two dimensions.

1 Introduction

Functional equations on hypergroups have been treated in [6, 7]. In this paper we study functional equations on a special hypergroup, which is related to the set of continuous unitary irreducible representations of the group $G = SU(2)$, the *special linear group* in two dimensions. We show how to determine all exponentials, additive

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The research was supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. NK-81402 and TAMOP-4.2.2/B-10/1-2010-0024 project.

Keywords and phrases: functional equation, polynomial hypergroup

AMS (2000) Subject Classification: 20N20, 60F99
functions and generalized moment function sequences on this hypergroup. Moment
functions on other types of hypergroups have been described in [3], [4] and [5]. The
definition of the underlying hypergroup is taken from [1].

If $G$ is a compact topological group then its dual object $\hat{G}$ consists of equivalence
classes of continuous irreducible representations of $G$. For any two classes $U, V$ of
this type their tensor product can be decomposed into its irreducible components
$U_1, U_2, \ldots, U_n$ with the respective multiplicities $m_1, m_2, \ldots, m_n$ (see [2]). We define
convolution on $\hat{G}$ by

$$\delta_U \ast \delta_V = \sum_{i=1}^{n} \frac{m_i d(U_i)}{d(U) d(V)} \delta_{U_i}$$

where $d(U)$ denotes the dimension of $U$ and $\delta_U$ is the Dirac measure concentrated at
$U$. Then $\hat{G}$ with this convolution and with the discrete topology is a commutative
hypergroup.

In the special case of $G = SU(2)$ the dual object $\hat{G}$ can be identified with the set $\mathbb{N}$
of natural numbers as it is indicated in [1]: the set of equivalence classes of continuous
unitary irreducible representations of $SU(2)$ is given by \{\(T^{(0)}, T^{(1)}, T^{(2)}, \ldots\)\}, where
$T^{(n)}$ has dimension $n + 1$, and we identify this set with $\mathbb{N}$.

For every $m, n$ in $\mathbb{N}$ the tensor product of $T^{(m)}$ and $T^{(n)}$ is unitary equivalent to

$$T^{(|m-n|)} \bigoplus T^{(|m-n|+2)} \bigoplus \ldots \bigoplus T^{(m+n)}.$$  \hspace{1cm} (2)

The convolution is given by

$$\delta_m \ast \delta_n = \sum_{k=|m-n|}^{m+n} \frac{k+1}{(m+1)(n+1)} \delta_k,$$ \hspace{1cm} (3)

where the prime denotes that every second term appears in the sum, only. With this
convolution $\mathbb{N}$ becomes a discrete commutative hypergroup, and since all the $T^{(n)}$
are self-conjugate, the hypergroup is in fact Hermitian. We call this hypergroup the \textit{SU}(2)-hypergroup.

2 Exponential functions on the \textit{SU}(2)-hypergroup

In this section we describe the exponential functions on the \textit{SU}(2)-hypergroup. We recall that the function \( M : \mathbb{N} \rightarrow \mathbb{C} \) is an exponential if and only if it satisfies

\[ M(m)M(n) = M(m * n) = \sum_{k=|m-n|}^{m+n} \frac{k+1}{(m+1)(n+1)} M(k) \]  

for all natural numbers \( m, n \).

**Theorem 1.** The function \( M : \mathbb{N} \rightarrow \mathbb{C} \) is an exponential on the \textit{SU}(2)-hypergroup if and only if there exists a complex number \( \lambda \) such that

\[ M(n) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh \lambda} \]  

holds for each natural number \( n \). (Here \( \lambda = 0 \) corresponds to the exponential \( M = 1 \).)

**Proof.** Let \( M : \mathbb{N} \rightarrow \mathbb{C} \) be a solution of (4) and let \( f(n) = (n+1)M(n) \) for each \( n \) in \( \mathbb{N} \). Then we have

\[ f(m)f(n) = \sum_{k=|m-n|}^{m+n} f(k) \]

for each \( m, n \) in \( \mathbb{N} \). With \( m = 1 \) it follows that \( f \) satisfies the following second order homogeneous linear difference equation

\[ f(n+2) - f(1)f(n+1) + f(n) = 0 \]  

for each \( n \) in \( \mathbb{N} \) with \( f(0) = 1 \).

Suppose that \( f(1) = 2 \). Then from (6) we infer that \( f(n) = n + 1 \) and \( M = 1 \) which corresponds to the case \( \lambda = 0 \) in (5). Otherwise \( f(1) \neq 2 \) and let \( \lambda \neq 0 \) be a
complex number with \( f(1) = 2 \cosh \lambda \). Then we have that

\[
f(n) = a e^{n \lambda} + \beta e^{-n \lambda}
\]

holds for any \( n \in \mathbb{N} \) with some complex numbers \( a, \beta \) satisfying \( a + \beta = 1 \). It is easy to see that in this case

\[
f(n) = \frac{\sinh[(n+1)\lambda]}{\sinh \lambda}
\]

holds for each \( n \in \mathbb{N} \). Finally, we have

\[
M(n) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh \lambda}.
\]

Conversely, it is easy to check that any function \( M \) of the given form is an exponential on the \( SU(2) \)-hypergroup, hence the theorem is proved.

\[\square\]

## 3 Additive functions on the \( SU(2) \)-hypergroup

Now we describe the additive functions on the \( SU(2) \)-hypergroup. We recall that the function \( A : \mathbb{N} \to \mathbb{C} \) is an additive function if and only if it satisfies

\[
A(m) + A(n) = A(m \ast n) = \sum_{k=|m-n|}^{m+n} \frac{k + 1}{(m+1)(n+1)} A(k)
\]

for all natural numbers \( m, n \).

**Theorem 2.** The function \( A : \mathbb{N} \to \mathbb{C} \) is an additive function on the \( SU(2) \)-hypergroup if and only if there exists a complex number \( c \) such that

\[
A(n) = \frac{c}{3} n(n+2)
\]

holds for each natural number \( n \).

**Proof.** Let \( A : \mathbb{N} \to \mathbb{C} \) be a solution of (7) and let \( f(n) = (n+1)A(n) \) for each \( n \) in \( \mathbb{N} \). Then we have

\[
(n+1)f(m) + (m+1)f(n) = \sum_{k=|m-n|}^{m+n} \frac{f(k)}{k+1}
\]
for each $m, n$ in $\mathbb{N}$. With $m = 1$ it follows that $f$ satisfies the following second order homogeneous linear difference equation

$$f(n + 2) - 2f(n + 1) + f(n) = 2c(n + 2)$$

for each $n$ in $\mathbb{N}$ with $f(0) = 0$ and $f(1) = 2c$. As the second difference of $f$ is linear it follows that $f$ is a cubic polynomial and simple computation gives that $A$ has the desired form.

Conversely, it is easy to check that any function $A$ of the given form is an additive function on the $SU(2)$-hypergroup, hence the theorem is proved.

\[ \square \]

## 4 Generalized moment functions on the $SU(2)$-hypergroup

Finally we describe the generalized moment functions on the $SU(2)$-hypergroup. Let $N$ be a nonnegative integer. We recall that the functions $\varphi_0, \varphi_1, \ldots, \varphi_N : \mathbb{N} \to \mathbb{C}$ form a generalized moment function sequence if and only if they satisfy

$$\varphi_k(m \ast n) = \sum_{j=0}^{k} \binom{k}{j} \varphi_j(m) \varphi_{k-j}(n)$$

(8)

for all natural numbers $m, n$ and for $k = 0, 1, \ldots, N$.

Making use of the results in Section 2 we introduce the function

$$\Phi(n, \lambda) = \frac{\sinh((n + 1)\lambda)}{(n + 1) \sinh \lambda}$$

(9)

for each $n$ in $\mathbb{N}$ and $\lambda \neq 0$ in $\mathbb{C}$, while $\Phi(n, 0) = 1$ for each $n$ in $\mathbb{N}$. The function $\Phi : \mathbb{N} \times \mathbb{C} \to \mathbb{C}$ is called an exponential family for the $SU(2)$-hypergroup: each exponential on this hypergroup has the form $n \mapsto \Phi(n, \lambda)$ with some unique $\lambda$ in $\mathbb{C}$,
and, conversely, the function \( n \mapsto \Phi(n, \lambda) \) is an exponential on the \( SU(2) \)-hypergroup for every complex \( \lambda \).

**Theorem 3.** Let \( K \) denote the \( SU(2) \)-hypergroup and \( \Phi \) the exponential family given by \( (9) \). The functions \( \varphi_0, \varphi_1, \ldots, \varphi_N : K \to \mathbb{C} \) form a generalized moment sequence of order \( N \) on \( K \) if and only if there exist complex numbers \( c_j \) for \( j = 1, 2, \ldots, N \) such that

\[
\varphi_k(n) = \frac{d^k}{dt^k} \Phi(n, f(t))(0)
\]

holds for each \( n \) in \( \mathbb{N} \) and for \( k = 0, 1, \ldots, N \), where

\[
f(t) = \sum_{j=0}^{N} c_j t^j
\]

for each \( t \) in \( \mathbb{C} \).

**Proof.** First we note that, by \( (3) \), we have for \( n \geq 1 \)

\[
\delta_n * \delta_1 = \sum_{k=n-1}^{n+1} \delta_k = \frac{k+1}{2(n+1)} \delta_k = \frac{n}{2(n+1)} \delta_{n-1} + \frac{n+2}{2(n+1)} \delta_{n+1},
\]

hence, by 3.2.1 Proposition in \( [1] \), \( K \) is a polynomial hypergroup, that is, there exists a sequence \( (P_n)_{n \in \mathbb{N}} \) of polynomials such that \( \deg P_n = n \) for \( n = 0, 1, \ldots \), there exists an \( x_0 \) in \( \mathbb{R} \) such that \( P_n(x_0) = 1 \) for \( n = 0, 1, \ldots \), and

\[
P_n(x)P_m(x) = \sum_{k=0}^{\infty} c(m, n, k) P_k(x)
\]

for each \( x \) in \( \mathbb{R} \) and \( m, n \) in \( \mathbb{N} \) with some nonnegative numbers \( c(m, n, k) \), further we have

\[
\delta_m * \delta_n = \sum_{k=0}^{\infty} c(m, n, k) \delta_k
\]

for each \( m, n \) in \( \mathbb{N} \). Here we shall determine this sequence of polynomials.

Our basic observation is that the function \( \lambda \mapsto \Phi(n, \lambda) \) is a polynomial of \( \cosh \lambda \) of degree \( n \) for each \( n \) in \( \mathbb{N} \). We apply mathematical induction. For \( n = 0 \) and \( n = 1 \),
we have by (9)
\[ \Phi(0, \lambda) = \frac{\sinh \lambda}{\sinh \lambda} = 1, \]
\[ \Phi(1, \lambda) = \frac{\sinh(2\lambda)}{2\sinh \lambda} = \cosh \lambda. \]

Suppose, that for \( k = 0, 1, \ldots, n \) there exists a polynomial \( P_k \) of degree \( k \) such that
\[ \Phi(k, \lambda) = P_k(\cosh \lambda) \] (14)
holds. Clearly \( P_0(x) = 1 \) and \( P_1(x) = x \). Then, by equation (11), we have
\[ P_n(\cosh \lambda) \cosh \lambda = \frac{n}{2(n + 1)} P_{n-1}(\cosh \lambda) + \frac{n + 2}{2(n + 1)} \Phi(n + 1, \lambda), \]
that is
\[ \Phi(n + 1, \lambda) = \frac{2(n + 1)}{n + 2} P_n(\cosh \lambda) \cosh \lambda - \frac{n}{n + 2} P_{n-1}(\cosh \lambda), \]
and here the right hand side is a polynomial of degree \( n + 1 \) in \( \cosh \lambda \):
\[ P_{n+1}(x) = \frac{2(n + 1)}{n + 2} x P_n(x) - \frac{n}{n + 2} P_{n-1}(x), \]
hence
\[ \Phi(n + 1, \lambda) = P_{n+1}(\cosh \lambda), \]
which was to be proved.

Finally, we have for all \( m, n \) in \( \mathbb{N} \) and \( \lambda \) in \( \mathbb{C} \)
\[ P_n(\cosh \lambda)P_m(\cosh \lambda) = \Phi(n, \lambda)\Phi(m, \lambda) = \Phi(n + m, \lambda) = \]
\[ = \sum_{k = |m - n|}^{m+n} \frac{k + 1}{(m + 1)(n + 1)} \Phi(k, \lambda) = \sum_{k = |m - n|}^{m+n} \frac{k + 1}{(m + 1)(n + 1)} P_k(\cosh \lambda), \]
which implies
\[ P_n(x)P_m(x) = \sum_{k = |m - n|}^{m+n} \frac{k + 1}{(m + 1)(n + 1)} P_k(x) \]
for each \( x \) in \( \mathbb{R} \) and \( m, n \) in \( \mathbb{N} \). This means that \( K \) is the polynomial hypergroup associated to the sequence of polynomials \( (P_n)_{n \in \mathbb{N}} \). Then, by Theorem 4. in [4], our statement follows.

\[ \square \]
References


