FUNCTIONAL EQUATIONS ON HYPERGROUPS

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Abstract. This paper presents some recent results concerning functional equations on hypergroups. The aim is to give some idea for the treatment of classical functional equation problems in the hypergroup setting. The general form of additive functions, exponentials and moment functions of second order on discrete polynomial hypergroups is given. In addition, stability problems for additive and exponential functions on hypergroups are considered.

1. Introduction

The concept of DJS–hypergroup (according to the initials of C. F. Dunkl, R. I. Jewett and R. Spector) is due to R. Lasser (see e.g. [4]). One begins with a locally compact Haussdorff space K, the space $\mathcal{M}(K)$ of all finite complex regular measures on K, the space $\mathcal{M}_c(K)$ of all finitely supported measures in $\mathcal{M}(K)$, the space $\mathcal{M}^1(K)$ of all probability measures in $\mathcal{M}(K)$, and the space $\mathcal{M}_c^1(K)$ of all compactly supported probability measures in $\mathcal{M}(K)$. The point mass concentrated at x is denoted by δ_x. Suppose that we have the following:

(H^*) There is a continuous mapping $(x,y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $\mathcal{M}_c^1(K)$, the latter being endowed with the weak*-topology with respect to the space of compactly supported complex valued continuous functions on K. This mapping is called *convolution*.
($H^\vee$) There is an involutive homeomorphism $x \mapsto x^\vee$ from K to K. This mapping is called *involution*.
(He) There is a fixed element e in K. This element is called *identity*.

Identifying x by δ_x the mapping in (H^*) has a unique extension to a continuous bilinear mapping from $\mathcal{M}(K) \times \mathcal{M}(K)$ to $\mathcal{M}(K)$. The involution on K extends to an involution on $\mathcal{M}(K)$. Then a *DJS–hypergroup*, or simply *hypergroup* is a quadruple $(K, *, \vee, e)$ satisfying the following axioms: for any x, y, z in K we have

($H1$) $\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z$;
($H2$) $(\delta_x * \delta_y)^\vee = \delta_{y^\vee} * \delta_{x^\vee}$;
($H3$) $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;
($H4$) e is in the support of $\delta_x * \delta_{y^\vee}$ if and only if $x = y$;
($H5$) the mapping $(x,y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space of nonvoid compact subsets of K is continuous, the latter being endowed with the Michael–topology (see [1]).

If $\delta_x * \delta_y = \delta_y * \delta_x$ holds for all x, y in K, then we call the hypergroup *commutative*. If $x^\vee = x$ holds for all x in K then we call the hypergroup *Hermitian*. By (H2)

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any Hermitian hypergroup is commutative. For instance, if $K = G$ is a locally compact Haussdorff–group, $\delta_x * \delta_y = \delta_{xy}$ for all x, y in K, $x^\vee$ is the inverse of x, and e is the identity of G, then we obviously have a hypergroup $(K, *, \vee, e)$, which is commutative if and only if the group G is commutative. The same works if $K = S$ is a locally compact Haussdorff–semigroup with identity, and involution is the identity mapping. However, not every hypergroup originates in this way, as it is shown by the following example.

Let $0 < \theta \leq 1$ be arbitrary and let $K = \{0, 1\}$. We define e as 0 and involution as the identity map. The products $\delta_0 * \delta_0 = \delta_0$, $\delta_0 * \delta_1 = \delta_1 * \delta_0 = \delta_1$ are obvious, and we let

$$\delta_1 * \delta_1 = \theta\delta_0 + (1 - \theta)\delta_1.$$

It is easy to see that we get a hypergroup for any θ in $]0, 1]$. In fact, any two-element hypergroup arises in this way. For $\theta = 1$ we get the two-element group of integers modulo 2.

In any hypergroup K we identify x by δ_x and we define the *right translation operator* T_y by the element y in K according to the formula:

$$T_y f(x) = \int_K f \, d(\delta_x * \delta_y),$$

for any f integrable with respect to $\delta_x * \delta_y$. In particular, T_y is defined for any continuous complex valued function on K. Similarly, we can define *left translation operators* but at this moment we do not need any extra notation for them.

Sometimes one uses the suggestive notation

$$f(x * y) = \int_K f \, d(\delta_x * \delta_y),$$

for any x, y in K. However, we call the attention to the fact that actually $f(x * y)$ has no meaning in itself, because $x * y$ is in general not an element of K, hence f is not defined at $x * y$. The expression $x * y$ denotes a kind of "indistinct" product. If B is a Borel–subset of K, then $\delta_x * \delta_y(B)$, or, what is the same, $x * y(B)$ expresses the probability of the event that the "product" of x and y belongs to the set B. In the special case of groups, or semigroups this probability is 1, if B contains xy, and is 0 otherwise, that is, exactly $\delta_{xy}(B)$.

If K is a discrete topological space, then we call the hypergroup a *discrete hypergroup*. As in this paper we deal mainly with discrete hypergroups we summarize here its axioms.

Let K be a set and suppose that the following properties are satisfied.

(D^*) There is a mapping $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $\mathcal{M}_c^1(K)$, the space of all finitely supported probability measures on K endowed with the weak*-topology with respect to the space of finitely supported complex valued functions on K. This mapping is called *convolution*.

($D^\vee$) There is an involutive bijection $x \mapsto x^\vee$ from K to K. This mapping is called *involution*.

(De) There is a fixed element e in K. This element is called *identity*.

Identifying x by δ_x as above convolution extends to a continuous bilinear mapping from $\mathcal{M}(K) \times \mathcal{M}(K)$ to $\mathcal{M}(K)$, and the involution K extends to an involution on $\mathcal{M}(K)$. Then a *discrete DJS–hypergroup* is a quadruple $(K, *, \vee, e)$ satisfying the following axioms: for any x, y, z in K we have

$(D1)$ $\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z$;
$(D2)$ $(\delta_x * \delta_y)^{\vee} = \delta_{y^{\vee}} * \delta_{x^{\vee}}$;
$(D3)$ $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;
$(D4)$ e is in the support of $\delta_x * \delta_{y^{\vee}}$ if and only if $x = y$.

2. Polynomial hypergroups

An important special class of Hermitian hypergroups is closely related to orthogonal polynomials.

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be real sequences with the following properties: $c_n > 0$, $b_n \geq 0$, $a_{n+1} > 0$ for all n in $\mathbb{N}$, moreover $a_0 = 0$, and $a_n + b_n + c_n = 1$ for all n in $\mathbb{N}$. We define the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ by $P_0(x) = 1$, $P_1(x) = x$, and by the recursive formula

$$xP_n(x) = a_n P_{n-1}(x) + b_n P_n(x) + c_n P_{n+1}(x)$$

for all $n \geq 1$ and x in $\mathbb{R}$. The following theorem holds.

Theorem 2.1. *If the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ satisfies the above conditions, then there exist constants $c(n, m, k)$ for all n, m, k in $\mathbb{N}$ such that*

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) P_k$$

holds for all n, m in $\mathbb{N}$.

Proof. By the theorem of Favard (see [2], [5]) the conditions on the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ imply that there exists a probability measure μ on $[-1, 1]$ such that $(P_n)_{n\in\mathbb{N}}$ forms an orthogonal system on $[-1, 1]$ with respect to μ. As P_n has degree n, we have

$$P_n P_m = \sum_{k=0}^{n+m} c(n, m, k) P_k$$

for all n, m in $\mathbb{N}$, where

$$c(n, m, k) = \frac{\int_{-1}^{1} P_k P_n P_m \, d\mu}{\int_{-1}^{1} P_k^2 \, d\mu}$$

holds for all n, m, k in $\mathbb{N}$. The orthogonality of $(P_n)_{n\in\mathbb{N}}$ with respect to μ implies $c(n, m, k) = 0$ for $k > n + m$ or $n > m + k$ or $m > n + k$. Hence our statement is proved.

The formula in the theorem is called *linearization formula*, and the coefficients $c(n, m, k)$ are called *linearization coefficients*. The recursive formula for the sequence $(P_n)_{n\in\mathbb{N}}$ implies $P_n(1) = 1$ for all n in $\mathbb{N}$, hence we have

$$\sum_{k=|n-m|}^{n+m} c(n, m, k) = 1$$

for all n in $\mathbb{N}$. It may or may not happen that $c(n,m,k) \geq 0$ for all n,m,k in $\mathbb{N}$. If it happens then we can define a hypergroup structure on $\mathbb{N}$ by the following rule:

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)\delta_k$$

for all n,m in $\mathbb{N}$, with involution as the identity mapping and with e as 0. The resulting discrete Hermitian (hence commutative) hypergroup is called *the polynomial hypergroup associated with the sequence* $(P_n)_{n\in\mathbb{N}}$.

As an example we consider the hypergroup associated with the Legendre–polynomials. The corresponding recurrence relation is

$$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$$

for all $n \geq 1$ and x in $\mathbb{R}$. It can be seen easily that the linearization coefficients are nonnegative, and the resulting hypergroup associated with the Legendre–polynomials is the *Legendre–hypergroup*.

3. Exponential polynomials on hypergroups

In the case of commutative groups exponential polynomials play a fundamental role in several problems concerning functional equations. As exponential polynomials are built up from additive and exponential functions, which are closely related to translation operators, the presence of translation operators on hypergroups makes it possible to define these basic functions on hypergroups, too.

Let K be a hypergroup with convolution $*$, involution $^\vee$, and identity e. For any y in K let T_y denote the right translation operator on the space of all complex valued functions on K which are integrable with respect to $\delta_x * \delta_y$ for any x,y in K. In particular, any continuous complex valued function belongs to this class. We call the continuous complex valued function a on K *additive*, if it satisfies

$$T_y a(x) = a(x) + a(y)$$

for all x,y in K. In more details this means that

$$\int_K a(t)\, d(\delta_x * \delta_y)(t) = a(x) + a(y)$$

holds for any x,y in K. The continuous complex valued function m on K is called an *exponential*, if it is not identically zero, and

$$T_y m(x) = m(x)m(y)$$

holds for all x,y in K. In other words m satisfies the functional equation

$$\int_K m(t)\, d(\delta_x * \delta_y)(t) = m(x)m(y).$$

It is obvious that any linear combination of additive functions is additive again. However, in contrast with the case of groups, the product of exponentials is not necessarily an exponential. The bounded exponential m is called a *character*, if $m(x^\vee) = \overline{m(x)}$ holds for any x in K. Obviously $a(e) = 0$ for any additive function a, and $m(e) = 1$ for any exponential m.

An *exponential monomial* on a locally compact Abelian group G is a product of additive and exponential functions, that is, a function of the form

$$x \mapsto a_1(x)a_2(x)\dots a_n(x)m_1(x)m_2(x)\dots m_k(x),$$

where n, k are nonnegative integers, $a_1, a_2, \dots, a_n : G \to \mathbb{C}$ are additive functions, and $m_1, m_2, \dots, m_k$ are exponentials. As the product of exponentials is an exponential again, we can write any exponential monomial in the form

$$x \mapsto a_1(x)a_2(x)\dots a_n(x)m(x),$$

where n is a nonnegative integer, $a_1, a_2, \dots, a_n : G \to \mathbb{C}$ are additive functions, and m is an exponential. In the case $n = 0$ we consider this function to be identically equal to m. A linear combination of exponential monomials is called an *exponential polynomial*. A product of additive functions is called a *monomial*, and a product of monomials we call a *polynomial*.

If we want to introduce these concepts on commutative, or arbitrary hypergroups, we have to remember the fact that product of exponentials is not necessarily an exponential. On the other hand in case of commutative groups exponential polynomials can be characterized by the fact that the linear space of functions spanned by the translates is finite dimensional. This property is of fundamental importance from the point of view of spectral synthesis. Hence it seems to be reasonable to define exponential polynomials by this property, even on arbitrary - not necessarily commutative - hypergroups. Anyway, additive and exponential functions on hypergroups obviously have this property and it seems to be interesting to describe these function classes on different hypergroups.

4. Exponential functions on polynomial hypergroups

Let $K = (\mathbb{N}, *)$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. Now we describe all exponential functions defined on K (see also [1]).

Theorem 4.1. *Let K be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The function $\varphi : \mathbb{N} \to \mathbb{C}$ is an exponential on K if and only if there exists a complex number z such that*

$$\varphi(n) = P_n(z)$$

holds for all n in $\mathbb{N}$.

Proof. First of all we remark that if a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ satisfies a recursion of the form

$$P_n(x)P_m(x) = \sum_{k=0}^{n+m} c(n, m, k)P_k(x)$$

with some real or complex coefficients $c(n,m,k)$ for all real x, then the recursion holds for all complex x. Let z be a complex number and $\varphi(n) = P_n(z)$ for any n in $\mathbb{N}$. Then by the definition of convolution we have for any m, n in $\mathbb{N}$

$$\varphi(\delta_n * \delta_m) = \sum_{k=|n-m|}^{n+m} c(n,m,k)\varphi(k) =$$

$$= \sum_{k=|n-m|}^{n+m} c(n,m,k)P_k(z) = P_n(z)P_m(z) = \varphi(n)\varphi(m),$$

hence φ is exponential.

Conversely, let φ be an exponential on K and we define $z = \varphi(1)$. By the exponential property we have for all positive integer n that

$$z\varphi(n) = \varphi(1)\varphi(n) = \varphi(\delta_1 * \delta_n) = \sum_{k=n-1}^{n+1} c(n,1,k)\varphi(k) =$$

$$= c(n,1,n-1)\varphi(n-1) + c(n,1,n)\varphi(n) + c(n,1,n+1)\varphi(n+1).$$

As the same recursion holds for $n \mapsto P_n(z)$, further $\varphi(0) = 1 = P_0(z)$ and $\varphi(1) = z = P_1(z)$, hence $\varphi(n) = P_n(z)$ for all n in $\mathbb{N}$ and the theorem is proved.

Hence any exponential function on a polynomial hypergroup arises from the evaluations $n \mapsto P_n(z)$ associated with the generating polynomials.

5. Additive functions on polynomial hypergroups

Again we let $K = (\mathbb{N}, *)$ the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. We describe all additive functions defined on K (see also [1]).

Theorem 5.1. *Let K be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The function $a : \mathbb{N} \to \mathbb{C}$ is an additive function on K if and only if there exists a complex number c such that*

$$a(n) = cP_n'(1)$$

holds for all n in $\mathbb{N}$.

Proof. Suppose first that $a : \mathbb{N} \to \mathbb{C}$ is an additive function on K. Then by the additive property we have for any positive integer n that

$$a(n) + a(1) = a(\delta_n * \delta_1) = \sum_{k=n-1}^{n+1} c(n,1,k)a(k) =$$

$$= c(n,1,n-1)a(n-1) + c(n,1,n)a(n) + c(n,1,n+1)a(n+1).$$

On the other hand, we have for any positive integer n and for all real x that

$$xP(x) = c(n,1,n-1)P_{n-1}(x) + c(n,1,n)P_n(x) + c(n,1,n+1)P_{n+1}(x).$$

Differentiating both sides with respect to x and substituting $x = 1$ we get

$$1 + P_n'(1) = c(n,1,n-1)P_{n-1}'(1) + c(n,1,n)P_n'(1) + c(n,1,n+1)P_{n+1}'(1)$$

for any positive integer n. Multiplying both sides by $a(1)$ we see that the functions $n \mapsto a(n)$ and $n \mapsto a(1)P_n'(1)$ satisfy the same recursion, further $a(0) = P_0'(1)$ and $a(1) = a(1)P_1'(1)$, hence $a(n) = a(1)P_n'(1)$ for all n in $\mathbb{N}$.

Conversely, we consider the linearization formula

$$P_n(x)P_m(x) = \sum_{k=|n-m|}^{n+m} P_k(x),$$

which holds for any n, m in $\mathbb{N}$ and for any real x. Differentiating both sides with respect to x and substituting $x = 1$ we get

$$P_n(1)P_m'(1) + P_n'(1)P_m(1) = \sum_{k=|n-m|}^{n+m} c(n,m,k)P_k'(1)$$

for all n, m in $\mathbb{N}$. As $P_n(1) = P_m(1) = 1$ for all n, m in $\mathbb{N}$, this formula shows that the function $n \mapsto P_n'(1)$ is additive, hence $n \mapsto cP_n'(1)$ is additive for any complex number c, and the theorem is proved.

For instance, in the case of the Tchebycheff–hypergroup of the first kind one gets the general form of additive functions as

$$a(n) = cn^2$$

for all n in $\mathbb{N}$, where c is an arbitrary complex number.

6. Moment functions on polynomial hypergroups

Let K be a hypergroup and N a positive integer. The continuous function $\varphi : K \to \mathbb{C}$ is called a *moment function of order* N, if there are complex valued continuous functions $\varphi_k : K \to \mathbb{C}$ for $k = 0, 1, \ldots, N$ such that $\varphi_0 = 1$, $\varphi_N = \varphi$, and

$$\varphi_k(\delta_x * \delta_y) = \sum_{j=0}^{k} \binom{k}{j} \varphi_j(x)\varphi_{n-j}(y)$$

holds for $k = 0, 1, \ldots, N$ and for all x, y in K. In this case we say that the functions φ_k $(k = 0, 1, \ldots, N)$ form *a moment sequence of order* N. Hence moment functions of order 1 are exactly the additive functions. The study of moment functions and moment sequences on hypergroups leads to the study of the above system of functional equations. In case of polynomial hypergroups we can describe the moment sequences of order N completely. Here we give the result for $N = 2$ only; the general result will be published elsewhere. We need the following simple theorem.

Theorem 6.1. *Let K be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. If the function $\psi : \mathbb{N} \to \mathbb{C}$ satisfies*

$$\psi(\delta_n * \delta_1) = \psi(n) + \psi(1)$$

for all n in $\mathbb{N}$, then ψ is additive.

Proof. In the proof we will use the following notation: for any n in $\mathbb{N}$ let $p(n) = P_n'(1)$.

By differentiating the linearization formula and substituting $x = 1$ we have

$$p(\delta_n * \delta_1) = p(n) + 1,$$

hence

$$\psi(1)p(\delta_n * \delta_1) = \psi(1)p(n) + \psi(1)$$

holds for all n in $\mathbb{N}$. As $\psi(0) = p(0) = 0$ and $\psi(1) = \psi(1)p(1)$, and the functions $n \mapsto \psi(n)$ and $n \mapsto \psi(1)p(n)$ satisfy the same recurrence relation of order 2, they must be equal: $\psi(n) = \psi(1)P_n'(1)$. The theorem is proved.

Theorem 6.2. *Let K be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The complex valued functions $\varphi_0, \varphi_1, \varphi_2$ on K form a moment sequence if and only if*

$$\varphi_0(n) = 1,$$
$$\varphi_1(n) = aP_n'(1),$$
$$\varphi_2(n) = a^2 P_n''(1) + bP_n'(1),$$

for all n in $\mathbb{N}$, where a, b are arbitrary complex numbers.

Proof. We will use the following notation: for any n in $\mathbb{N}$ and for $k = 0, 1, 2$ let $p_k(n) = P_n^{(k)}(1)$.

By differentiating k times the linearization formula for the sequence $(P_n)_{n\in\mathbb{N}}$ ($k = 1, 2$) and substituting $x = 1$ we have

$$p_1(\delta_n * \delta_m) = p_1(n) + p_1(m),$$

and

$$p_2(\delta_n * \delta_m) = p_2(n) + 2p_1(n)p_1(m) + p_2(m)$$

for all n, m in $\mathbb{N}$. Using these relations an easy computation shows that the functions φ_k ($k = 0, 1, 2$) given in the theorem form a moment sequence of order 2 with arbitrary complex constants a, b.

Conversely, suppose now that the functions φ_k ($k = 0, 1, 2$) given in the theorem form a moment sequence of order 2. Then we have that φ_1 is additive and by the previous theorem $\varphi_1(n) = aP_n'(1)$ for all n in $\mathbb{N}$, with $a = \varphi_1(1)$. On the other hand for any n in $\mathbb{N}$ it follows

$$\varphi_2(\delta_n * \delta_1) = \varphi_2(n) + 2a\varphi_1(n) + \varphi_2(1),$$

and from the above equation with $m = 1$

$$p_2(\delta_n * \delta_1) = p_2(n) + 2p_1(n).$$

Hence we have the following two equations for all n in $\mathbb{N}$:

$$\varphi_2(\delta_n * \delta_1) = \varphi_2(n) + 2a^2 p_1(n) + \varphi_2(1),$$
$$a^2 p_2(\delta_n * \delta_1) = a^2 p_2(n) + 2a^2 p_1(n).$$

By subtraction we have that the function $\psi : \mathbb{N} \to \mathbb{C}$ defined on $\mathbb{N}$ by

$$\psi(n) = \varphi_2(n) - a^2 p_2(n)$$

satisfies the relation

$$\psi(\delta_n * \delta_1) = \psi(n) + \psi(1)$$

for all n in $\mathbb{N}$. By the previous theorem ψ is additive, and by Theorem 5.1. we get $\psi(n) = bP_n'(1)$ for all n in $\mathbb{N}$ with some complex number b. By substitution we have

$$\varphi_2(n) = a^2 P_n''(1) + bP_n'(1)$$

for all n in $\mathbb{N}$, and the theorem is proved.

7. Stability problems on hypergroups

Stability theory of functional equations on groups, on semigroups and on different algebraic structures has attracted the attention of several mathematicians recently. In this respect see [8]. According to our knowledge stability problems for functional equations on hypergroups have not been considered so far. In this section we show that using similar ideas as in the case of groups, semigroups, etc., analogous results can be obtained.

First we deal with the stability of exponential functions on hypergroups. The following result is similar to that of [7].

Theorem 7.1. *Let K be a hypergroup and let $f, g : K \to \mathbb{C}$ be continuous functions with the property that the function $y \mapsto \int_K f\, d(\delta_x * \delta_y) - f(x)g(y)$ is bounded for all y in K. Then either f is bounded, or g is exponential.*

Proof. It is easy to see ([1]) that for any continuous function $\varphi : K \to \mathbb{C}$ and for any y in K the functions $x \mapsto \int_K \varphi\, d(\delta_x * \delta_y)$ and $y \mapsto \int_K \varphi\, d(\delta_x * \delta_y)$ are continuous. First of all we can check easily that the associativity of the convolution implies that

$$\int_K \int_K \varphi(s)\, d(\delta_t * \delta_z)(s)\, d(\delta_x * \delta_y)(t) = \int_K \int_K \varphi(s)\, d(\delta_x * \delta_t)(s)\, d(\delta_y * \delta_z)(t)$$

holds for all x, y, z in K. Actually on the left hand side we have

$$\int_K \varphi\, d[(\delta_x * \delta_y) * \delta_z],$$

and on the right hand side

$$\int_K \varphi\, d[\delta_x * (\delta_y * \delta_z)].$$

The assumption of the theorem means that

$$\Big|\int_K f(s)\,d(\delta_t*\delta_z)-f(t)g(z)\Big|\leq M(z)$$

holds for all x,y in K with some given continuous function $M:K\to\mathbb{C}$. Integrating with respect to the measure $\delta_x*\delta_y$ and with respect to the variable t we have for all x,y,z in K:

$$\Big|\int_K\int_K f(s)\,d(\delta_t*\delta_z)(s)\,d(\delta_x*\delta_y)(t)-\int_K f(t)g(z)\,d(\delta_x*\delta_y)(t)\Big|\leq M(z).$$

Let

$$A(x,y,z)=\int_K\int_K f(s)\,d(\delta_t*\delta_z)(s)\,d(\delta_x*\delta_y)(t)-\int_K f(t)g(z)\,d(\delta_x*\delta_y)(t)$$

for all x,y,z.

Using again the assumption of the theorem we have

$$\Big|\int_K f(s)\,d(\delta_x*\delta_t)(s)-f(x)g(t)\Big|\leq M(t)$$

for all x,t in K. Now we integrate both sides with respect to the measure $\delta_y*\delta_z$ and with respect to the variable t we get

$$\Big|\int_K\int_K f(s)\,d(\delta_x*\delta_t)(s)\,d(\delta_y*\delta_z)(t)-\int_K f(x)g(t)\,d(\delta_y*\delta_z)(t)\Big|\leq$$

$$\leq\int_K M(t)\,d(\delta_y*\delta_z)(t)$$

for all x,y,z in K. Let

$$B(x,y,z)=\int_K\int_K f(s)\,d(\delta_x*\delta_t)(s)\,d(\delta_y*\delta_z)(t)-\int_K f(x)g(t)\,d(\delta_y*\delta_z)(t)$$

for all x,y,z in K.

Our next observation is that

$$\Big|f(x)g(y)g(z)-\int_K f(t)\,d(\delta_x*\delta_y)(t)g(z)\Big|\leq g(z)M(y)$$

holds for any x,y,z in K. Let

$$C(x,y,z)=f(x)g(y)g(z)\int_K f(t)\,d(\delta_x*\delta_y)(t)g(z)$$

for all x,y,z in K.

Finally, we consider the identity

$$A(x,y,z)-B(x,y,z)-C(x,y,z)=f(x)\Big(\int_K g(t)\,d(\delta_y*\delta_z)(t)-g(y)g(z)\Big),$$

which holds for all x, y, z in K. From the above inequalities we can see that the left hand side is bounded for any fixed y, z in K, hence if g is not an exponential, then f is bounded, and the theorem is proved.

The stability of additive functions on hypergroups can be derived easily in the presence of an invariant mean, which depends on the hypergroup. However, an invariant mean always exists on commutative hypergroups, as it follows from the Markov–Kakutani fixed point theorem (see [3]). We recall that an *invariant mean* M on a commutative discrete hypergroup K is a bounded linear functional on the Banach space of all bounded complex valued functions defined on K with the property that $M(1) = 1$, and

$$M_x\Big[\int_K \varphi\, d(\delta_x * \delta_y)\Big] = M(\varphi)$$

for any bounded function $\varphi : K \to \mathbb{C}$ and for any y in K.

Theorem 7.2. *Let K be a commutative discrete hypergroup and $f : K \to \mathbb{C}$ a function with the property that the function*

$$(x, y) \mapsto \int_K f\, d(\delta_x * \delta_y) - f(x) - f(y)$$

is bounded on $K \times K$. Then there exists an additive function $a : K \to \mathbb{C}$ such that $f - a$ is bounded.

Proof. We can prove this theorem following the lines of [7]. For any fixed y in K we define

$$a(y) = M_x\big[\int_K f\, d(\delta_x * \delta_y) - f(x)\big],$$

where M is any invariant mean on K, and M_x indicates that we apply M to the (obviously bounded) function in the brackets, as a function of x. Now we have

$$\int_K a(t)\, d(\delta_y * \delta_z)(t) - a(y) - a(z) =$$

$$= M_x\Big[\int_K \int_K f(s)\, d(\delta_z * \delta_t)(s)\, d(\delta_y * \delta_x)(t) - \int_K f(t)\, d(\delta_x * \delta_y)(t)\Big] -$$

$$-M_x\Big[\int_K f(t)\, d(\delta_x * \delta_z)(t) - f(x)\Big]$$

for all y, z in K. If $\varphi : K \to \mathbb{C}$ is the function defined by

$$\varphi(x) = \int_K f(t)\, d(\delta_x * \delta_z)(t) - f(x)$$

for all x, z in K, then the above equation can be written in the form

$$\int_K a(t)\, d(\delta_y * \delta_z)(t) - a(y) - a(z) = M_x\Big[\varphi(t)\, d(\delta_x * \delta_y)(t)\Big] - M(\varphi),$$

and the right hand side is 0 by the invariance of M. It means that a is additive. On the other hand,

$$|a(y) - f(y)| = \left| M_x \left[\int_k f(t)\, d(\delta_x * \delta_y)(t) - f(x) - f(y) \right] \right| \leq C$$

holds for all y in K, where C is a bound for the function

$$(x, y) \mapsto \int_K f\, d(\delta_x * \delta_y) - f(x) - f(y).$$

The theorem is proved.

References

1. Bloom, W.R., Heyer, H., *Harmonic Analysis of Probability Measures on Hypergroups*, Berlin, New York, 1995.
2. Favard, J., *Sur les polynomes de Tchebycheff*, C. R. Acad. Sci. Paris **200(A-B)** (1935), 2052-2053.
3. Lau, A.T., *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups*, Fund. Math. **118** (1983), 161-175.
4. Ross, K.A., *Hypergroups and Signed Hypergroups*, Harmonic Analysis on Hypergroups (Anderson, J.M., Litvinov, G.L., Ross, K.A., Singh, A.I., Sunder, V.S., Wildberger, N.J., eds.), de Gruyter Studies in Mathematics, de Gruyter, Birkhäuser, Boston, Basel, Berlin, 1998, pp. 77-91.
5. Shohat, J., *The relation of the classical orthogonal polynomials to the polynomials of Appel*, Amer. J. Math. (1936), 453-464.
6. Székelyhidi, L., *On a theorem of Baker, Lawrence and Zorzitto*, Proc. Amer. Math. Soc. **84** (1982), 95-96.
7. Székelyhidi, L., *Remark 17.*, Aequationes Math. **29** (1982).
8. Székelyhidi, L., *Ulam's problem, Hyers's solution - and to where they led*, Stability of functional equations (Rassias, Th.M., eds.), Kluwer Academic Publishers, Boston, Dordrecht, London, 2000, pp. 259-285.