# FUNCTIONAL DIFFERENTIAL EQUATIONS BY SPLINE FUNCTIONS

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## 1. Introduction

In this paper we present a new method for the approximate solution of functional differential equations. The method is based on third-order spline functions. Suppose that a partition is given on an interval on which we are to solve a first order system of functional differential equations. At the knots we construct approximate values of the solution successively in the following way: the next value is obtained by substituting a third order spline function into the right side of the equation, which takes the previous approximate values at the previous knots, and then integrating it on the next interval. We prove convergence theorems concerning our method and the stability is also established.

Our method can be applied in very general situations whenever the existence and uniqueness of the solution is assured. Namely, we impose a Lipschitz condition on the right side and the continuity of the given functions is assumed.

As we have simple recursive formulae for the approximate values, our method can easily be programmed and the computations can be implemented on small calculators. We illustrate the method by a numerical example. We proceeded the calculations by an electronic calculator TI$-$59.

Concerning other numerical methods for solving functional differential equations based on spline functions see [1].

Throughout this paper **R** denotes the set of real numbers. If $B$ is a Banach-space and $x:[a, b]\to B$ continuous or continuously differentiable function, then $\omega(h, x)$ and $\omega_1(h, x)$ denotes the modulus of continuity of $x$ and $x'$, respectively.

## 2. Construction of the spline function

Let $[a, b]\subset\mathbf{R}$ be an interval and $B$ a Banach-space. Fix a subdivision $a = t_0 < t_1 < \ldots < t_N = b$ of $[a, b]$ and let $h_K = t_K - t_{K-1}$ $(K = 1, \ldots, N)$, further $h = \max_{1\le K\le N} h_K$, $\mu = h/\min_{1\le K\le N} h_K$. If $x:[a, b]\to B$ is a continuously differentiable function, then we denote $x_K = x(t_K)$ $(K = 0, 1, \ldots, N)$.

Define the spline function $S$ on $[a, b]$ as follows:

$$(1) \qquad S(t) = S_K(t) \quad (t_{K-1} \le t \le t_K),$$

where

$$S_1(t) = x_1 + \frac{1}{h_1}(x_1 - x_0)(t - t_1)$$

and for $K = 2, \ldots, N$

$$S_K(t) = x_K + \frac{1}{h_K}(x_K - x_{K-1})(t - t_K) -$$

$$-\frac{1}{h_K}\left[\frac{1}{h_K}(x_K - x_{K-1}) - \frac{1}{h_{K-1}}(x_{K-1} - x_{K-2})\right]\left[(t-t_K)^2 + \frac{1}{h_K}(t-t_K)^3\right].$$

It is obvious that $S$ is continuously differentiable on $[a, b]$.

**THEOREM 2.1.** *For the spline function $S$ defined by* (1) *we have for all $t$ in $[a, b]$:*

$$\|x(t) - S(t)\| \le 6h\,\omega_1(h, x),$$

$$\|x'(t) - S'(t)\| \le 12\,\omega_1(h, x). \boxtimes$$

**Proof.** Let $t_{K-1} \le t \le t_K$ $(K = 1, \ldots, N)$, then

$$x(t) = x_K + x'(\tau_K)(t - t_K),$$

where $t < \tau_K < t_K$, further let

$$x_K^*(t) = x_K + x_K'(t - t_K),$$

where $x_K' = x'(t_K)$. Then using the Lagrange theorem, we have

$$\|x(t) - S(t)\| \le \|x(t) - x_K^*(t)\| + \|x_K^*(t) - S(t)\| \le \|x'(\tau_K) - x_K'\|\, h_K +$$

$$+ \left\|x_K' - \frac{1}{h_K}(x_K - x_{K-1})\right\| h_K + 2h_K\left\|\frac{1}{h_K}(x_K - x_{K-1}) - \frac{1}{h_{K-1}}(x_{K-1} - x_{K-2})\right\| \le$$

$$\le 6h\,\omega_1(h, x),$$

and

$$\|x'(t) - S'(t)\| \le \|x'(t) - x_K'\| + \|x_K' - S'(t)\| \le \|x'(t) - x_K'\| +$$

$$+ \left\|x_K' - \frac{1}{h_K}(x_K - x_{K-1})\right\| + 5\left\|\frac{1}{h_K}(x_K - x_{K-1}) - \frac{1}{h_{K-1}}(x_{K-1} - x_{K-2})\right\| \le$$

$$\le 12\omega_1(h, x). \boxtimes$$

**THEOREM 2.2.** *Let $x_K, \tilde{x}_K$ in $B$ be given and $\|x_K-\tilde{x}_K\| \le \varepsilon$ $(K=0,1,\dots,N)$. Let $S$ and $\tilde{S}$ denote the spline function of the form (1) with the values $x_K$, $\tilde{x}_K$. Then we have for all $t$ in $[a,b]$:*

$$\|S(t)-\tilde{S}(t)\| \le (7+4\mu)\,\varepsilon\,,$$

$$\|S'(t)-\tilde{S}'(t)\| \le 22\,\frac{\mu}{h}\,\varepsilon\,. \boxtimes$$

**Proof.** Let $t_{K-1} \le t \le t_K$ $(K = 1, \dots, N)$, then

$$\|S(t)-\tilde{S}(t)\| = \|S_K(t)-\tilde{S}_K(t)\| \le \|x_K-\tilde{x}_K\| + \|(x_K-x_{K-1})-(\tilde{x}_K-\tilde{x}_{K-1})\| +$$

$$+2\|(x_K-x_{K-1})-(\tilde{x}_K-\tilde{x}_{K-1})\| + 2\,\frac{h_K}{h_{K-1}}\|(x_{K-1}-x_{K-2})-(\tilde{x}_{K-1}-\tilde{x}_{K-2})\| \le$$

$$\le (7+4\mu)\,\varepsilon\,,$$

and

$$\|S'(t)-\tilde{S}'(t)\| = \|S'_K(t)-\tilde{S}'_K(t)\| \le \frac{1}{h_K}\,\|(x_K-x_{K-1})-(\tilde{x}_K-\tilde{x}_{K-1})\| +$$

$$+\frac{5}{h_K}\,\|(x_K-x_{K-1})-(\tilde{x}_K-\tilde{x}_{K-1})\| + \frac{5}{h_{K-1}}\,\|(x_{K-1}-x_{K-2})-(\tilde{x}_{K-1}-\tilde{x}_{K-2})\| \le$$

$$\le 22\,\frac{\mu}{h}\,\varepsilon\,. \boxtimes$$

**REMARK 2.3.** In case of equidistant partition we have $\mu = 1$ and the last theorem gives the estimates

$$\|S(t)-\tilde{S}(t)\| \le 11\,\varepsilon\,,$$

$$\|S'(t)-\tilde{S}'(t)\| \le 22\,\frac{\varepsilon}{h}\,. \boxtimes$$

## 3. Approximate solution of functional differential equations

Let $t_0$ be a real number, $0<\gamma<\delta$, further let $B$ be a Banach-space, and $\Theta : [t_0-\delta, t_0] \to B$, $f : [t_0, \infty)\times B^{n+1} \to B$, $r_i : [t_0, \infty) \to [\gamma, \delta]$ $(i = 1, \dots, n)$ be continuous functions where $f$ satisfies the Lipschitz condition

$$\|f(t, y_1, \dots, y_{n+1})-f(t, \overline{y}_1, \dots, \overline{y}_{n+1})\| \le L \sum_{j=1}^{n+1} \|y_j-\overline{y}_j\|\,,$$

whenever $t \ge t_0$ and $y_j$, $\overline{y}_j$ are in $B$ $(j = 1, \dots, n+1)$.

Suppose that the function $x : [t_0-\delta, \infty) \to B$ satisfies the system of equations

$$x'(t) = f\bigl(t, x(t), x(t-r_1(t)), \dots, x(t-r_n(t))\bigr) \quad (t \ge t_0), \tag{2}$$
$$x(t) = \Theta(t) \quad (t_0-\delta \le t \le t_0).$$

For $T > t_0$ let us consider the partition

$$t_0-\delta = t_{-M} < t_{-M+1} < \dots < t_{-1} < t_0 < t_1 < \dots < t_N = T$$

of $[t_0-\delta, T]$, where $t_K - t_{K-1} = h_K < \gamma$ $(K = -M+1, \dots, N)$. Denote $S$ the spline function of the form (1) with the values $x_K = x(t_K)$. Then $S$ is an approximate solution of (2). For if $t_0-\delta \le t \le t_0$, then

$$\|S(t)-\Theta(t)\| = \|S(t)-x(t)\| \le 6h\,\omega_1(h, x),$$

and if $t \ge t_0$, then

$$\Bigl\|S'(t)-f\bigl(t, S(t), S(t-r_1(t)), \dots, S(t-r_n(t))\bigr)\Bigr\| \le$$
$$\le \Bigl\|S'(t)-x'(t)\Bigr\| + \Bigl\|f\bigl(t, x(t), x(t-r_1(t)), \dots, x(t-r_n(t))\bigr) -$$
$$- f\bigl(t, S(t), S(t-r_1(t)), \dots, S(t-r_n(t))\bigr)\Bigr\| \le$$
$$\le 12\omega_1(h, x) + L\sum_{j=0}^{n} \|x(t-r_j(t)) - S(t-r_j(t))\| \le$$
$$\le 12\omega_1(h, x) + 6L(n+1)\,h\,\omega_1(h, x),$$

where $r_0(t) \equiv 0$. Notice, that we know the values $x_K$ only for $K \le 0$, hence we cannot compute the coefficients of $S$.

In the next step we construct the approximate values $\tilde{x}_K$ $(K = -M, -M+1, \dots, N)$ and we show that the spline function $\tilde{S}$ with these values is an approximate solution of (2) and provides a good approximation for $x$.

Let $\tilde{x}_K = \Theta(t_K)$ for $K = -M, -M+1, \dots, 0$ and

$$\tilde{x}_{K+1} = \tilde{x}_K + \int_{t_K}^{t_{K+1}} f\bigl(t, \tilde{x}_K + \tilde{x}'_K(t-t_K), \tilde{S}_{(K)}(t-r_1(t)), \dots, \tilde{S}_{(K)}(t-r_n(t))\bigr)\,dt \tag{3}$$

for $K = 0, 1, \dots, N-1$, where

$$\tilde{x}'_K = f\bigl(t_K, \tilde{x}_K, \tilde{S}_{(K)}(t_K-r_1(t_K)), \dots, \tilde{S}_{(K)}(t_K-r_n(t_K))\bigr) \quad (K \ge 0),$$
$$\tilde{S}_{(0)}(t) = \Theta(t) \quad (t_0-\delta \le t \le t_0)$$

and $\tilde{S}_{(K)}$ $(K \ge 1)$ denotes the spline function of the form (1) with the values $\tilde{x}_j$ $(j = -M, \dots, K)$ on the interval $[t_0-\delta, t_K]$. Using (2) we have

$$x_{K+1} = x(t_{K+1}) = x_K + \int_{t_K}^{t_{K+1}} f\bigl(t, x(t), x(t-r_1(t)), \dots, x(t-r_n(t))\bigr)\,dt$$

and by the Lagrange theorem it follows for $t_K \le t \le t_{K+1}$ that $x(t) = x_K + x'(\tau_K)(t-t_K)$ where $t_K < \tau_K < t_{K+1}$. Let $S_{(K)}$ denote the spline function of the form (1) with the values $x_j$ ($j = -M, \ldots, K$) on the interval $[t_0-\delta, t_K]$. Then we have

$$\|x_{K+1}-\tilde{x}_{K+1}\| \le \|x_K-\tilde{x}_K\| +$$

$$+\int_{t_K}^{t_{K+1}} \left\|f\big(t, x(t), \ldots, x(t-r_n(t))\big) - f\big(t, \tilde{x}_K + \tilde{x}'_K(t-t_K), \ldots, \tilde{S}_{(K)}(t-r_n(t))\big)\right\| dt \le$$

$$\le \|x_K-\tilde{x}_K\| + L\int_{t_K}^{t_{K+1}} \|x(t)-\tilde{x}_K-\tilde{x}'_K(t-t_K)\|\,dt +$$

$$+L\sum_{j=1}^{n}\int_{t_K}^{t_{K+1}} \|x(t-r_j(t)) - \tilde{S}_{(K)}(t-r_j(t))\| \le \|x_K-\tilde{x}_K\| +$$

$$+L\int_{t_K}^{t_{K+1}} [\|x_K-\tilde{x}_K\| + \|x'(\tau_K)-x'(t_K)\|\,(t-t_K) + \|x'(t_K)-\tilde{x}'_K\|\,(t-t_K)]\,dt +$$

$$+L\sum_{j=1}^{n}\int_{t_K}^{t_{K+1}} [\|x(t-r_j(t)) - S_{(K)}(t-r_j(t))\| + \|S_{(K)}(t-r_j(t)) - \tilde{S}_{(K)}(t-r_j(t))\|]\,dt\,.$$

Now let

$$\delta_K = \max_{0\le j\le K} \|x_j-\tilde{x}_j\|\,,$$

then

$$\|x_{K+1}-\tilde{x}_{K+1}\| \le \|x_K-\tilde{x}_K\| + L\,h_{K+1}\|x_K-\tilde{x}_K\| + L\,\frac{h^2_{K+1}}{2}\,\omega_1(h,x) +$$

$$+L\,\frac{h^2_{K+1}}{2}\left\|f\big(t_K, x_K, \ldots, x(t_K-r_n(t_K))\big) - f\big(t_K, \tilde{x}_K, \ldots, \tilde{S}_{(K)}(t_K-r_n(t_K))\big)\right\| +$$

$$+6L\,h\,h_{K+1}\,n\,\omega_1(h,x) + (7+4\mu)\,L\,h_{K+1}\,n\,\delta_K \le$$

$$\le \|x_K-\tilde{x}_K\|\,(1+Lh) + \frac{L}{2}\,h^2\,\omega_1(h,x) +$$

$$+\frac{L^2}{2}\,h^2\Big[\|x_K-\tilde{x}_K\| + \sum_{j=1}^{n}\|x(t_K-r_j(t_K)) - S_{(K)}(t_K-r_j(t_K))\| +$$

$$+\sum_{j=1}^{n}\|S_{(K)}(t_K-r_j(t_K)) - \tilde{S}_{(K)}(t_K-r_j(t_K))\|\Big] +$$

$$+6L\,h^2\,n\,\omega_1(h,x) + (7+4\mu)\,L\,h\,n\,\delta_K\,.$$

This implies

$$\|x_{K+1}-\tilde{x}_{K+1}\| \le \delta_K(1+c_0) + c_1\,h^2\,\omega_1(h,x)\,,$$

where $c_0$ and $c_1$ are constants independent of $h$. We remark that these constants are dependent on $\mu$. But if we suppose that $\mu$ is bounded from above by a constant independent of $h$, then the latter inequalities give us

$$\delta_{K+1} \leq c_2 h\, \omega_1(h, x),$$

where the constant $c_2$ is independent of $h$ (see e.g. [2]).

**THEOREM 3.1.** *Suppose that the functions in* (2) *satisfy the above mentioned conditions. Then the spline function* $\tilde{S}$ *with the values* $\tilde{x}_K$ $(K = -M, \ldots, \ldots, N)$ *constructed by* (3) *has the properties*

$$\|x(t) - \tilde{S}(t)\| \leq \text{const. } h\, \omega_1(h, x)$$

$$\|x'(t) - \tilde{S}'(t)\| \leq \text{const. } \omega_1(h, x)$$

$$\|\tilde{S}'(t) - f\big(t, \tilde{S}(t), \tilde{S}(t - r_1(t)), \ldots, \tilde{S}(t - r_n(t))\big)\| \leq \text{const. } \omega_1(h, x)$$

*for* $t > 0$ *and*

$$\|\Theta(t) - \tilde{S}(t)\| \leq \text{const. } h\, \omega_1(h, x)$$

*for* $-\delta \leq t \leq 0$.

This theorem is an easy consequence of the previous considerations. We remark, that the constants here may depend on the number $\mu$. But if we let $h \to 0$ and $\mu$ remains bounded (for instance $\mu = 1$, in case of equidistant partitions), our theorem gives the convergence of $\tilde{S}$ to the exact solution.

## 4. Application

Now we apply our previous results for the following linear problem

$$x'(t) = A_0 x(t) + \sum_{j=1}^{n} A_j x(t-j) + c \quad (t > 0) \tag{4}$$

$$x(t) = \Theta(t) \quad (t \leq 0)$$

where $A_j$ is a bounded linear operator of the Banach space $B$ $(j = 0, 1, \ldots, n)$, $\Theta : [-n, 0] \to B$ is a given function and $c$ is an element of $B$. In this case we obtain particularly simple recursive equations for the approximate values $\tilde{x}_K$.

Let $T > 0$ be an integer, $h = \dfrac{1}{N}$ and $t_K = \dfrac{K}{N}$ $(K = -Nn, \ldots, NT)$.

By (3) we have

$$\tilde{x}_K = x_K = \Theta(Kh)$$

for $K = -Nn, \ldots, -1, 0$, and

$$\tilde{x}_1 = \tilde{x}_0'(I + A_0 h) + A_0 \tilde{x}_0' \frac{h^2}{2} + \sum_{j=1}^{n} A_j \int_{-j}^{-j+h} \Theta(t)\, dt + ch,$$

where

$$\tilde{x}'_0 = \sum_{j=0}^{n} A_j \Theta(-j) + c$$

($I$ denotes the identity operator). Further, for $k \geq 1$

$$\tilde{x}_{K+1} = \tilde{x}_K + \int_{t_K}^{t_{K+1}} A_0(\tilde{x}_K + \tilde{x}'_K(t - t_K))\,dt + \sum_{j=1}^{n} A_j \int_{t_K}^{t_{K+1}} \tilde{S}_{(K)}(t-j)\,dt + ch,$$

where

$$\tilde{x}'_K = \sum_{j=0}^{n} A_j \tilde{x}_{K-jN} + c.$$

On the other hand, an easy computation gives

$$\int_{t_K}^{t_{K+1}} \tilde{S}_{(K)}(t-j)\,dt = \int_{t_{K-jN}}^{t_{K+1-jN}} \tilde{S}_{(K)}(t)\,dt =$$

$$= h\left[\frac{5}{12}\tilde{x}_{K+1-jN} + \frac{2}{3}\tilde{x}_{K-jN} - \frac{1}{12}\tilde{x}_{K-1-jN}\right].$$

Finally, we have the following recursive formulae for the problem (4):

(5)

$$\tilde{x}_K = \Theta(Kh) \quad (K = -Nn, \ldots, -1, 0),$$

$$\tilde{x}_1 = (I + A_0 h)\tilde{x}_0 + \frac{h^2}{2}\sum_{j=0}^{n} A_0 A_j \Theta(-j) + \frac{h^2}{2} A_0 c + hc + \sum_{j=1}^{n} A_j \int_{-j}^{-j+h} \Theta(t)\,dt,$$

$$\tilde{x}_{K+1} = \left(I + A_0 h + A_0^2 \frac{h^2}{2}\right)\tilde{x}_K + \frac{h^2}{2}\left(\sum_{j=1}^{n} A_0 A_j \tilde{x}_{K-jN} + A_0 c\right) +$$

$$+ hc + h\sum_{j=1}^{n} A_j\left[\frac{5}{12}\tilde{x}_{K+1-jN} + \frac{2}{3}\tilde{x}_{K-jN} - \frac{1}{12}\tilde{x}_{K-1-jN}\right] \; (K = 1, 2, \ldots, NT).$$

## 5. Example

Here we consider the example

$$x'(t) = 5x(t) + x(t-1) \quad (t > 0)$$

$$x(t) = 5 \qquad (-1 \leq t \leq 0)$$

(see [1]). The exact solution on [0, 1] is

$$x(t) = 6e^{5t} - 1$$

and on [1, 2] is

$$x(t) = \left[x(1) - \frac{1}{5} + 6(t-1)\right] e^{5(t-1)} + \frac{1}{5}.$$

Using $N = 900$ we obtain the results summarized in the following table:

*Table*

| $t$ | $x(t)$ | $\tilde{S}(t)$ | $x(t)-\tilde{S}(t)$ |
|---|---|---|---|
| 0. | 5. | 5. | 0. |
| 0.1 | 8.892327624 | 8.892302286 | 0.000025338 |
| 0.2 | 15.30969097 | 15.30960742 | 0.00008355 |
| 0.3 | 25.89013442 | 25.88992779 | 0.00020663 |
| 0.4 | 43.33433659 | 43.33388236 | 0.00045423 |
| 0.5 | 72.09496376 | 72.09402765 | 0.00093611 |
| 0.6 | 119.5132215 | 119.5113695 | 0.001852 |
| 0.7 | 197.6927118 | 197.6891493 | 0.0035625 |
| 0.8 | 326.5889002 | 326.5821876 | 0.0067126 |
| 0.9 | 539.1027878 | 539.0903372 | 0.0124506 |
| 1.0 | 889.4789546 | 889.4561617 | 0.0227929 |
| 1.1 | 1467.362361 | 1467.326525 | 0.035836 |
| 1.2 | 2420.772761 | 2420.730253 | 0.042508 |
| 1.3 | 3993.738812 | 3993.673437 | 0.065375 |
| 1.4 | 6588.865818 | 6588.765777 | 0.100041 |
| 1.5 | 10870.38298 | 10870.23076 | 0.15222 |
| 1.6 | 17934.15321 | 17933.92318 | 0.23003 |
| 1.7 | 29588.1594 | 29587.81453 | 0.34487 |
| 1.8 | 48815.2569 | 48814.74486 | 0.51204 |
| 1.9 | 80536.63293 | 80535.88165 | 0.75128 |
| 2.0 | 132871.3779 | 132870.292 | 1.0859 |

## REFERENCES


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[2] *Th. Fawzy*, Spline Functions and the Cauchy Problems, III., Annales Univ. Sci. B dapest., Sectio Computatorica 1 (1978), 35–45.