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(1) C.R. Math, Rep., Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1.

(2) Prof. P. Ribenboim, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L 3N6.

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# Equations arising from the theory of orthogonally additive and quadratic functions


By Gy. Maksa, Gy. Szabó and L. Székelyhidi

*Presented by J. Aczél, F.R.S.C.*


Throughout this paper[1], $(\mathcal{F},+,\cdot)$ and $(\mathcal{G},+),(\mathcal{S},+)$ denote a commutative field and two abelian groups, respectively. The study of orthogonally additive resp. quadratic mappings on abstract orthogonality spaces (see [2] and some forthcoming papers of the second author) leads to the following unrestricted equations for the unknown functions $g,h:\mathcal{F}\to\mathcal{S}$, respectively:

$$(1)\quad g(ax+by)-g(ax)-g(by)=g(ay+bx)-g(ay)-g(bx),\qquad x,y\in\mathcal{F},$$

$$(2)\quad h(ax+by)+h(ax-by)-2h(ax)-2h(by)= \\ =h(ay+bx)+h(ay-bx)-2h(ay)-2h(bx),\qquad x,y\in\mathcal{F},$$

where $a,b\in\mathcal{F}$ are fixed elements such that $a,b,a\pm b\neq 0$. In fact, (1) is a consequence of the more complex equation with two unknown functions $f,g:\mathcal{F}\to\mathcal{S}$, as follows

$$(3)\quad f(a_1x+b_1y)+g(a_2x+b_2y)= \\ =f(a_1x)+f(b_1y)+g(a_2x)+g(b_2y),\qquad x,y\in\mathcal{F},$$

where $a_1,a_2,b_1,b_2\in\mathcal{F}\setminus\{0\}$ are fixed elements such that $a_1b_2\neq a_2b_1$. Our aim with this paper is to solve these equations under fairly weak conditions on $\mathcal{F}$ and $\mathcal{S}$. We shall make use of the following result, which somewhat generalizes that of Székelyhidi (see [3] and also [1]; here for $p\in\mathbf{N}$, a group is *p-divisible*, *p-torsion-free* or *uniquely p-divisible*, if the $p$-multiplication on the group is surjective, injective or bijective, respectively):

[1] Research work of the third author has been supported by the Alexander von Humbold Foundation.

**THEOREM 1** *Let $n \in \mathbb{N}$ and suppose that $\mathcal{G}$ or $\mathcal{S}$ is $n!$-divisible and $\mathcal{S}$ is also $n!$-torsion-free. If the function $P : \mathcal{G} \to \mathcal{S}$ is of degree $n$, i.e.*

$$P(x) + \sum_{k=0}^{n} P_k(\phi_k(x) + \psi_k(y)) = 0, \qquad x, y \in \mathcal{G}, \tag{4}$$

*with some functions $P_k : \mathcal{G} \to \mathcal{S}$ and homomorphism $\phi_k, \psi_k : \mathcal{G} \to \mathcal{G}$ such that $\phi_k(\mathcal{G}) \subset \psi_k(\mathcal{G})$ $(k = 0, 1, \dots, n)$, then it has the form*

$$P(x) = P(0) + \sum_{k=1}^{n} A_k(x, x, \dots, x), \qquad x \in \mathcal{G}, \tag{5}$$

*where $A_k : \mathcal{G}^k \to \mathcal{S}$ are symmetric k-additive functions $(k = 1, \dots, n)$. The converse implication also holds whenever $\mathcal{G}$ is $(n+1)!$-divisible.*

*Proof.* (4) $\Longrightarrow$ (5): Without loss of generality, we may assume that $P(0) = 0$. Also, it is clear from the proof of [3], Thm. 3.6 that $P$ is a polynomial of degree $n$ (no divisibility is required). Thus for $n!$-divisible $\mathcal{S}$, Thm. 3 in [1] completes the proof. On the other hand, when $\mathcal{G}$ is the $n!$-divisible group, the proof of the quoted Theorem 3 in [1] yields only that

$$rP(x) = \sum_{k=1}^{n} r^k A_k^*\left(\frac{x}{r}, \frac{x}{r}, \dots, \frac{x}{r}\right), \qquad x \in \mathcal{G}, \tag{6}$$

with some symmetric k-additive functions $A_k^* : \mathcal{G}^k \to \mathcal{S}$ $(k = 1, 2, \dots, n)$ and $r = n!(n-1)! \dots 2!$, where $x/r$ is a symbol for any element in $\mathcal{G}$ with $r(x/r) = x$. Next observe that for any $k \in \mathbb{N}$ and arbitrary $u_i, u_i' \in \mathcal{G}$ such that $ru_i = ru_i'$ $(i = 1, 2, \dots, k)$,

$$r^k A_k^*(u_1, u_2, \dots, u_k) = r^k A_k^*(u_1', u_2', \dots, u_k')$$

holds. Thus since $\mathcal{S}$ is $n!$-torsion-free, the functions $A_k : \mathcal{G}^k \to \mathcal{S}$ $(k = 1, 2, \dots, n)$ are well defined by

$$A_k(x_1, x_2, \dots, x_k) = r^{k-1} A_k^*\left(\frac{x_1}{r}, \frac{x_2}{r}, \dots, \frac{x_k}{r}\right),$$

where $x_i \in \mathcal{G}$ and $x_i/r$ are arbitrary elements of $\mathcal{G}$ such that $r(x_i/r) = x_i$ $(i = 1, 2, \dots, k)$. Clearly $A_k$ is symmetric and also it is additive in each variable. This latter assertion immediately follows from the k-additivity of $A_k^*$ and the fact $r([x_i + x_i']/r) = r(x_i/r + x_i'/r)$. Thus (6) turns into

$$rP(x) = r \sum_{k=1}^{n} A_k(x, x, \dots, x), \qquad x \in \mathcal{G},$$



which since $\mathcal{S}$ is $n!$-torsion-free, proves the first implication.

(5) $\Longrightarrow$ (4) is immediate from the proof of [3], Thm. 3.6.

**COROLLARY 2** *Suppose that either $\operatorname{char}\mathcal{F} \neq 2$ and $\mathcal{S}$ is 2-torsion-free, or $\mathcal{S}$ is uniquely 2-divisible. Then $g$ is a solution of equation (1) if and only if, it is of the form*

$$g(x) = g(0) + A(x) + B(x, x), \qquad x \in \mathcal{F}, \tag{7}$$

*with an additive $A : \mathcal{F} \to \mathcal{S}$ and symmetric biadditive $B : \mathcal{F}^2 \to \mathcal{S}$ such that*

$$B(ax, by) = B(ay, bx), \qquad x, y \in \mathcal{F}. \tag{8}$$

*Proof. Necessity.* Introduce the new variables $u = ax + by$ and $v = ay + bx$. Then

$$x = \frac{au - bv}{a^2 - b^2}, \qquad y = \frac{av - bu}{a^2 - b^2}.$$

Thus (1) turns into the form

$$g(u) - g(v) + \left[g\left(\frac{b}{a^2 - b^2}[au - bv]\right) - g\left(\frac{a}{a^2 - b^2}[au - bv]\right)\right] +$$

$$+ \left[g\left(\frac{a}{a^2 - b^2}[-bu + av]\right) - g\left(\frac{b}{a^2 - b^2}[-bu + av]\right)\right] = 0, \qquad u, v \in \mathcal{F},$$

i.e. with suitably defined functions $g_k : \mathcal{F} \to \mathcal{S}$ $(k = 0, 1, 2)$,

$$g(u) + g_0(0u + 1v) + g_1(au - bv) + g_2(-bu + av) = 0, \qquad u, v \in \mathcal{F}, \tag{9}$$

and so the above Theorem implies (7). Finally, (8) comes from equation (1).

*Sufficiency.* Obvious.

**COROLLARY 3** *Suppose that either $\operatorname{char}\mathcal{F} \neq 2, 3$ and $\mathcal{S}$ is 6-torsion-free, or $\mathcal{S}$ is uniquely 6-divisible. Then $h$ is a solution of equation (2) if and only if, it has the form*

$$h(x) = h(0) + B(x, x) + D(x, x, x, x), \qquad x \in \mathcal{F}, \tag{10}$$

*with some symmetric 2-additive $B : \mathcal{F}^2 \to \mathcal{S}$ and a symmetric 4-additive function $D : \mathcal{F}^4 \to \mathcal{S}$ such that*

$$D(ax, ax, by, by) = D(ay, ay, bx, bx), \qquad x, y \in \mathcal{F}. \tag{11}$$

*Proof. Necessity.* Substitute in (2) $y = 0$. Then $h(bx) = h(-bx)$ for all $x \in \mathcal{F}$, i.e. $h$ is even. Now introduce the new variables $u = ax + by$ and $v = ay + bx$, whence

$$x = \frac{au - bv}{a^2 - b^2}, \qquad y = \frac{av - bu}{a^2 - b^2}.$$

Then (2) turns into the form

$$h(u) - h(v) + 2\left[h\left(\frac{b}{a^2 - b^2}[au - bv]\right) - h\left(\frac{a}{a^2 - b^2}[au - bv]\right)\right] +$$

$$+2\left[h\left(\frac{a}{a^2 - b^2}[-bu + av]\right) - h\left(\frac{b}{a^2 - b^2}[-bu + av]\right)\right] +$$

$$+h\left(\frac{a^2 + b^2}{a^2 - b^2}u - \frac{2ab}{a^2 - b^2}v\right) - h\left(\frac{-2ab}{a^2 - b^2}u + \frac{a^2 + b^2}{a^2 - b^2}v\right) = 0, \qquad u, v \in \mathcal{F},$$

i.e. with suitably defined functions $h_k : \mathcal{F} \to \mathcal{S}$ $(k = 0, 1, \ldots, 4)$ and elements $c = (a^2 + b^2)/(a^2 - b^2)$, $d = 2ab/(a^2 - b^2)$, we have

$$h(u) + h_0(0u + 1v) + h_1(au - bv) + h_2(-bu + av) + \tag{12}$$

$$+h_3(cu - dv) + h_4(-du + cv) = 0, \qquad u, v \in \mathcal{F}.$$

Thus by Theorem 1, $h$ is of form

$$h(x) = h(0) + \sum_{k=1}^{4} A_k(x, x, \ldots, x), \qquad x \in \mathcal{F},$$

with certain symmetric $k$-additive functions $A_k : \mathcal{F}^k \to \mathcal{S}$ $(k = 1, 2, 3, 4)$. Here, since $h$ is even and $\mathcal{S}$ is 6-torsion-free, we have for all $x \in \mathcal{F}$ that $A_1(x) + A_3(x, x, x) = 0$, whence $8A_3(x, x, x) = -2A_1(x)$ and therefore $4A_3(x, x, x) = -A_1(x) = A_3(x, x, x)$, i.e.

$$A_3(x, x, x) = 0 = A_1(x), \qquad x \in \mathcal{F}.$$

Finally, the formula (11) for $D = A_4$ can be verified by simple computations.

*Sufficiency.* Obvious.

In the rest of the paper we discuss equation (3) showing that the restrictions on $\mathcal{S}$ can be dropped as soon as the even and odd solutions are treated separately.



**THEOREM 4** *Assume that $char\mathcal{F} \neq 2$ and $f, g$ are solutions of equation (3). Then*

*i) $f$ or $g$ is odd if and only if, both of them are additive;*

*ii) $g$ is even if and only if, $g - g(0)$ is quadratic while $f$, with an additive $A : \mathcal{F} \to \mathcal{S}$ and $a = a_2/a_1$, $b = b_2/b_1$, has the form*

$$f(x) = f(0) + A(x) - g\left(\frac{a+b}{2}x\right) + g\left(\frac{a-b}{2}x\right), \qquad x \in \mathcal{F}. \tag{13}$$

*Proof.* By (3), $f(0) + g(0) = 0$ and so $f - f(0)$, $g - g(0)$ satisfy (3), too. Thus we may and do assume that $f(0) = g(0) = 0$. Now define $F, G : \mathcal{F}^2 \to \mathcal{S}$ by

$$F(u, v) = f(u + v) - f(u) - f(v), \qquad u, v \in \mathcal{F},$$

$$G(u, v) = g(u + v) - g(u) - g(v), \qquad u, v \in \mathcal{F}.$$

Then (3) turns into

$$F(u, v) + G(au, bv) = 0, \qquad u, v \in \mathcal{F}. \tag{14}$$

Hence, by the definition of $F$ and $G$, for all $u, v, w \in \mathcal{F}$, we have

$$G(au, bv) + G(au + bv, bw) = G(au, bv + bw) + G(bv, bw), \tag{15}$$

$$G(au, bv) + G(au + av, bw) = G(au, bv + bw) + G(av, bw). \tag{16}$$

Subtracting (15) from (16), it follows for all $u, v, w \in \mathcal{F}$ that

$$G(au + av, bw) - G(au + bv, bw) = G(av, bw) - G(bv, bw). \tag{17}$$

Since $G(0, bw) = 0$, therefore the substitution $u = -(b/a)v$ in (17) yields

$$G(av - bv, bw) = G(av, bw) - G(bv, bw), \qquad v, w \in \mathcal{F},$$

which implies by (17) that

$$G(au + av, bw) - G(au + bv, bw) = G(av - bv, bw), \qquad u, v, w \in \mathcal{F}. \tag{18}$$

Because of $char\mathcal{F} \neq 2$, this means that $G$ is additive in its first variable, and regarding the symmetry, $G$ is actually biadditive. Thus

$$g(u + v + w) - g(u + v) - g(w) = G(u + v, w) = G(u, w) + G(v, w) =$$

$$= g(u+w) - g(u) - g(w) + g(v+w) - g(v) - g(w), \qquad u,v,w \in \mathcal{F},$$

whence letting $w = -v$,

$$(19) \qquad g(u+v) + g(u-v) = 2g(u) + g(v) + g(-v), \qquad u,v \in \mathcal{F}.$$

In an analogous way, one can gain the same for $f$:

$$(20) \qquad f(u+v) + f(u-v) = 2f(u) + f(v) + f(-v), \qquad u,v \in \mathcal{F}.$$

Now if, say, $g$ is odd, then for $u = v$, (19) yields $g(2u) = 2g(u)$, whence

$$g(u+v) + g(u-v) = g(2u), \qquad u,v \in \mathcal{F},$$

i.e. regarding again the condition $char\mathcal{F} \neq 2$, $g$ is additive. By the original equation (3), so is $f$, proving part i).

Next suppose that $g$ is even. Then by (19), it is quadratic and due to the symmetry of $F$, for all $u, v \in \mathcal{F}$ we have $G(au, bv) = -F(u,v) = -F(v,u) = G(av, bu)$ and so

$$f(u+v) - f(u) - f(v) = -G(au, bv) = -2G\left(\frac{a}{2}u, \frac{b}{2}v\right) - 2G\left(\frac{a}{2}v, \frac{b}{2}u\right) =$$

$$= 2G\left(\frac{a}{2}u, \frac{b}{2}u\right) + 2G\left(\frac{a}{2}v, \frac{b}{2}v\right) - 2G\left(\frac{a}{2}[u+v], \frac{b}{2}[u+v]\right).$$

This means that the function $A : \mathcal{F} \to \mathcal{S}$, defined for each $x \in \mathcal{F}$ by

$$A(x) = f(x) + 2G\left(\frac{a}{2}x, \frac{b}{2}x\right) = f(x) + g\left(\frac{a+b}{2}x\right) - g\left(\frac{a-b}{2}x\right)$$

is additive, giving the formula (13).

The converse implications are obvious.

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KLTE Matem. Int., Pf. 12
H-4010 Debrecen, Hungary

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