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Spectral Synthesis and a Characterization of Polynomial Ideals

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Dedicated to the memory of Erdős Jenő

Abstract. The classical result about characterizing polynomial ideals in several variables by differential operators is the Ehrenpreis–Palamodov theorem. There are further results exhibiting a constructive method for finding the corresponding differential operators, the so-called Noetherian operators. Here we show the connection between this problem with discrete spectral synthesis and present a new method for constructing Noetherian operators for polynomial ideals.

1. Spectral synthesis on semigroups

The basic ideas of discrete spectral analysis and spectral synthesis can be formulated and investigated on commutative semigroups. Let S be a commutative semigroup written additively. The set of all complex valued functions on S will be denoted by $\mathcal{C}(S)$. This set equipped with the pointwise linear operations (addition and multiplication by complex numbers) and with the topology of pointwise convergence (the Thychonoff–topology) bears the structure of a locally convex topological vector space.

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For any function f on S having values in a set H the *translate of f by the element y in S* is the function $T_y f : S \to H$ defined by the equation

$$T_y f(x) = f(x + y)$$

for each x in S. A set of functions on S is called *translation invariant* if all translates of each function in the set belong to the set, too. Clearly any intersection of translation invariant sets is translation invariant. For a given set F of complex valued functions the intersection of all translation invariant sets including F is called *the translation invariant set generated by F*. For a given set F of complex valued functions on S the intersection of all translation invariant subspaces of $\mathcal{C}(S)$ including F is called *the translation invariant subspace generated by F*. Finally, for a given set F of complex valued functions on S the intersection of all translation invariant closed subspaces of $\mathcal{C}(S)$ including F is called *the variety generated by F*. This is obviously a translation invariant closed subspace of $\mathcal{C}(S)$, the smallest one of these properties, which includes F. In general, a *variety on S* is a closed translation invariant linear subspace of $\mathcal{C}(S)$. If F consists of a single function, say $F = \{f\}$, then the variety generated by F is called *the variety generated by f*. A nonzero variety is called *proper*. The statement that "the complex valued function g on S belongs to the variety generated by f" means that g is the pointwise limit of a net of functions, each of them being a linear combination of translates of f. Functions in the variety generated by f are exactly the ones which can be approximated in the sense of pointwise convergence by linear combinations of translates of f.

We remark that later we shall use the term "variety" with a different meaning in a different context.

The dual of $\mathcal{C}(S)$ can be identified with the space of all finitely supported complex Radon measures on G equipped with the weak*-topology and it is denoted by $\mathcal{M}_c(S)$. The pairing between $\mathcal{C}(S)$ and $\mathcal{M}_c(S)$ is given by

$$\langle f, \mu \rangle = \int f(x)\, d\mu(x)$$

for each f in $\mathcal{C}(S)$ and μ in $\mathcal{M}_c(S)$. The convolution between $\mathcal{C}(S)$ and

$\mathcal{M}_c(S)$, further between $\mathcal{M}_c(S)$ and $\mathcal{M}_c(S)$ is defined in the usual way (see e.g. [1]).

For each variety in $\mathcal{C}(S)$ the set of all elements in $\mathcal{M}_c(S)$ which are zero on the elements of the variety is called the *annihilator of the variety* and for any closed ideal in $\mathcal{M}_c(S)$ the set of all elements in $\mathcal{C}(S)$ on which every element of the ideal is zero is called the *annihilator of the ideal.* It is easy to see (see e.g. [5]) that the relationship between varieties in $\mathcal{C}(S)$ and closed ideals in $\mathcal{M}_c(S)$ is the following: the annihilator of any variety in $\mathcal{C}(S)$ is a closed ideal in $\mathcal{M}_c(S)$, which is proper if and only if the variety is proper. Conversely, the annihilator of any closed ideal in $\mathcal{M}_c(S)$ is a variety in $\mathcal{C}(S)$, which is proper if and only if the ideal is proper. Also, the annihilator of the annihilator is the original proper variety or proper closed ideal, respectively.

The basic building blocks of spectral synthesis are the additive and exponential functions, as well as exponential monomials and exponential polynomials. *Additive functions* on S are the homomorphisms of S into the additive group of complex numbers. *Polynomials* on S are obtained by substituting additive functions into complex polynomials in several variables. Hence the general form of a polynomial on S is the following: $x \mapsto P\big(a_1(x), a_2(x), \dots, a_n(x)\big)$, where P is a complex polynomial in n variables and $a_1, a_2, \dots, a_n$ are additive functions on S. *Exponential functions* on S are the non-identically zero homomorphisms of S into the multiplicative semigroup of complex numbers. If S is a group, then exponential functions cannot take the zero value. An *exponential monomial* on S is the product of a polynomial and an exponential function and linear combinations of exponential monomials are called *exponential polynomials.*

The basic question of *spectral analysis* is about the existence of an exponential function in a given proper variety. In the affirmative case we say that *spectral analysis holds for the given variety*, and if spectral analysis holds for each variety then we say that *spectral analysis holds in the semigroup.* The basic problem of *spectral synthesis* is if the exponential monomials in a given variety span a dense subvariety. In the affirmative case we say that *spectral synthesis holds for the given variety*, and if spectral synthesis holds for each variety then we say that *spectral synthesis holds in the semigroup*. For more about spectral analysis and spectral synthesis see e.g. [5] and the references in it. In this paper we will consider the special cases $S = \mathbb{Z}^n$ and $S = \mathbb{N}^n$ only.

Let n be a fixed positive integer. For each $z = (z_1, z_2, \ldots, z_n)$ in $\mathbb{C}^n$ and for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{N}^n$ we will use the notation $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}$ and $\alpha! = \alpha_1!\alpha_2! \ldots \alpha_n!$. If P is any complex polynomial in n variables, that is, any element of $\mathbb{C}[z] = \mathbb{C}[z_1, z_2, \ldots, z_n]$, the ring of all complex polynomials in n variables, then the notation for the differential operator $P(\partial) = P(\partial_1, \partial_2, \ldots, \partial_n)$ has the obvious meaning.

Using the simple ideas similar to those in the proofs of Theorem 6.10. and 6.11. in [5], p. 57. we have that any complex polynomial p on $\mathbb{Z}^n$ or on $\mathbb{N}^n$ is actually an ordinary complex polynomial in n variables and any exponential function m on $\mathbb{Z}^n$ or on $\mathbb{N}^n$ has the form

$$m(x_1, x_2, \ldots, x_n) = m_1(x_1)m_2(x_2)\ldots m_n(x_n)$$

for all $x_1, x_2, \ldots, x_n$ in $\mathbb{Z}$ or $\mathbb{N}$ with some exponentials of $\mathbb{Z}$ or $\mathbb{N}$ $(i = 1, 2, \ldots, n)$. However, in contrast to the case of $\mathbb{Z}$, on $\mathbb{N}$ we have a special exponential m_0, which is 1 for $x = 0$ and is 0 for $x \neq 0$. We shall use the notation $m_0(x) = 0^x$ for this exponential, which is correct if we agree on $0^0 = 1$. This means that the exponentials of $\mathbb{N}^n$ have the form

$$m(x_1, x_2, \ldots, x_n) = \lambda_1^{x_1} \lambda_2^{x_2} \ldots \lambda_n^{x_n}$$

for each $x_1, x_2, \ldots, x_n$ in $\mathbb{N}$ with arbitrary complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Hence the set of all exponentials of $\mathbb{N}^n$ can be identified with $\mathbb{C}^n$. We shall use the notation λ^x for the product $\lambda_1^{x_1} \lambda_2^{x_2} \ldots \lambda_n^{x_n}$ if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $x = (x_1, x_2, \ldots, x_n)$. For any finitely supported measure μ in $\mathcal{M}_c(\mathbb{N}^n)$ we will use its *Fourier–Laplace–transform* which is defined by

$$\widehat{\mu}(\lambda) = \int_{\mathbb{N}^n} \lambda^x \, d\mu(x)$$

for each λ in $\mathbb{C}^n$. This is a complex polynomial in n variables. Obviously any complex polynomial in n variables is the Fourier–Laplace–transform of some finitely supported measure on $\mathbb{N}^n$, hence the ring (actually algebra) of all Fourier–Laplace–transforms of finitely supported measures on $\mathbb{N}^n$

can be identified with the ring $\mathbb{C}[z_1, z_2, \ldots, z_n]$. Basically, the Fourier–Laplace–transformation $\mu \mapsto \widehat{\mu}$ identifies $\mathcal{M}_c(\mathbb{N}^n)$ with the polynomial ring $\mathbb{C}[z_1, z_2, \ldots, z_n]$. The exponential corresponding to λ belongs to a variety if and only if λ is a common root of the polynomials corresponding to the annihilator ideal of the variety. By Hilbert's Nullstellensatz the polynomials in any proper ideal in $\mathbb{C}[z_1, z_2, \ldots, z_n]$ have a common root (see e.g. [6]), thus we have the following result.

Theorem 1. *Spectral analysis holds in* $\mathbb{N}^n$.

It turns out that spectral synthesis also holds in $\mathbb{N}^n$. To verify this statement one needs the famous Lasker–Noether–theorem on primary decomposition (see [6]), which states that in $\mathbb{C}[z_1, z_2, \ldots, z_n]$ each proper ideal is the intersection of finitely many primary ideals. Using this theorem, a slight modification of the proof of Lefranc's theorem in [2] gives the following result.

Theorem 2. *Spectral synthesis holds in* $\mathbb{N}^n$.

2. Spectral synthesis and polynomial ideals

Characterization of polynomial ideals in several variables is the content of the Ehrenpreis–Palamodov theorem (see [4], Theorem 10.12., p. 141.). One of its consequences is the following theorem (see [4], Theorem 10.13., p. 142.).

Theorem 3. *Given any primary ideal I in the ring of complex polynomials in n variables there exist differential operators with polynomial coefficients*

$$A_i(x, \partial) = \sum_j p_j^i(x_1, x_2, \ldots, x_n)\partial_1^{j_1}\partial_2^{j_2}\ldots\partial_n^{j_n}$$

for $i = 1, 2, \ldots, r$ with the following property: a polynomial f lies in the ideal I if and only if the result of applying $A_i(x, \partial)$ to f vanishes on the (irreducible) variety of I for $i = 1, 2, \ldots, r$.

We recall that the *variety of a polynomial ideal* is the set of all common zeros of the polynomials in the ideal. The differential operators $A_1(x,\partial), A_2(x,\partial), \dots, A_r(x,\partial)$ are called *Noetherian operators* for the primary ideal I. An algorithm for computing Noetherian operators for a given primary ideal is given in [3]. Here we present another approach to this problem which is based on spectral synthesis on $\mathbb{N}^n$ and we present a simpler method for finding Noetherian operators.

We have seen in the previous section that the ring of complex polynomials $\mathbb{C}[z_1, z_2, \dots, z_n]$ can be identified with $\mathcal{M}_c(\mathbb{N}^n)$, the dual of $\mathcal{C}(\mathbb{N}^n)$, which is the space of all complex valued functions on $\mathbb{N}^n$ equipped with the topology of pointwise convergence. The weak*-topology on $\mathcal{M}_c(\mathbb{N}^n)$ is identical with the topology on $\mathbb{C}[z_1, z_2, \dots, z_n]$ corresponding to coefficient-wise convergence. Now we describe the identification between $\mathbb{C}[z_1, z_2, \dots, z_n]$ and $\mathcal{M}_c(\mathbb{N}^n)$ in more details.

Let p be a complex polynomial in $\mathbb{C}[z_1, z_2, \dots, z_n]$. Writing z for the vector $(z_1, z_2, \dots, z_n)$ the polynomial p can be written in the form

$$p(z) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial^\alpha p(0) z^\alpha$$

for all z in $\mathbb{C}^n$, where $\alpha! = \alpha_1!\alpha_2!\dots\alpha_n!$. Then the linear functional, or finitely supported measure μ_p, corresponding to p effects on a function f in $\mathcal{C}(\mathbb{N}^n)$ in the following way:

$$< \mu_p, f >= \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial^\alpha p(0) f(\alpha)\,.$$

Obviously, the convolution of μ_p and μ_r corresponds to $p\cdot r$. We observe that

$$\widehat{\mu_p}(\lambda) =< \mu_p(x), \lambda^x >= \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial^\alpha p(0)\lambda^x = p(\lambda),$$

hence the Fourier–Laplace–transform of μ_p can be identified with p. This means that we can simply write p for μ_p.

Theorem 4. *Let I be a primary ideal in the polynomial ring $\mathbb{C}[z_1, z_2, \dots, z_n]$. Then there exists a nonempty set $\mathcal{P}$ of linear differential operators with polynomial coefficients such that a polynomial p belongs to*

I if and only if

$$P(\xi, \partial)p(\xi) = 0 \tag{1}$$

holds for each ξ in the variety of I and for each P in $\mathcal{P}$.

(This means that the differential operators in $\mathcal{P}$ have the form as it was given in Theorem 3.)

PROOF. Let p be a polynomial in $\mathbb{C}[z_1, z_2, \dots, z_n]$. We have

$$p(\lambda) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha p(0) \lambda^\alpha \,,$$

hence

$$\partial^\beta p(\lambda) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha p(0) [\alpha]^\beta \lambda^{\alpha - \beta},$$

where $[\alpha]^\beta$ denotes the product $[\alpha_1]^{\beta_1} [\alpha_2]^{\beta_2} \dots [\alpha_n]^{\beta_n}$ with the usual notation $[\alpha_i]^{\beta_i} = \alpha_i(\alpha_i - 1) \dots (\alpha_i - \beta_i + 1)$ for $i = 1, 2, \dots, n$. It follows

$$\lambda^\beta \partial^\beta p(\lambda) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha p(0) [\alpha]^\beta \lambda^\alpha \,.$$

We consider an exponential monomial $\varphi : \mathbb{N}^n \to \mathbb{C}$ having the general form

$$\varphi(\alpha) = P(\alpha) \lambda^\alpha$$

with some complex polynomial P in n variables and λ in $\mathbb{C}^n$. The polynomial P has a unique representation in the form

$$P(\alpha) = \sum_{\beta \in \mathbb{N}^n} c_\beta [\alpha]^\beta \,,$$

which implies

$$\sum_{\beta \in \mathbb{N}^n} c_\beta \lambda^\beta \partial^\beta p(\lambda) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha p(0) P(\alpha) \lambda^\alpha \,.$$

Suppose that μ_p annihilates φ, that is $< \mu_p, \varphi >= 0$. This means

$$< \mu_p, \varphi >= \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha p(0) P(\alpha) \xi^\alpha = 0 \,,$$

or

$$\sum_{\beta\in\mathbb{N}^n} c_\beta \xi^\beta \partial^\beta p(\xi) = 0\,. \tag{2}$$

Suppose now that the primary ideal I is given and V is its irreducible variety, which is nonempty by Hilbert's Nullstellensatz (that is, by spectral analysis on $\mathbb{N}^n$.) By Theorem 2 the linear hull of all exponential monomials of the form φ is dense in the annihilator of I. This means that p belongs to the closure of I if and only if p satisfies a system of equations of the form (2), corresponding to the points ξ in the variety of I. To complete our proof it is enough to show that in the given topology on $\mathcal{M}_c(\mathbb{N}^n)$ any ideal is closed. It is obvious that the given topology is exactly the topology of coefficient-wise convergence of polynomials. On the other hand, if I is any proper ideal, then it is finitely generated, by Hilbert's Basis Theorem (see e.g. [6]). Let $f_1, f_2, \dots, f_N$ be a Groebner–basis of I with respect to any fixed monomial ordering and let R denote the operator on $\mathbb{C}[z_1, z_2, \dots, z_n]$ mapping each polynomial to its remainder with respect to the given Groebner–basis. Then R is linear and its kernel is exactly I. On the other hand, analyzing the division algorithm, it is clear that R is continuous with respect to the topology of coefficient-wise convergence: the coefficients of the remainder are continuous functions of the coefficients of the original polynomial. It follows that the kernel of R is closed and our theorem is proved. □

Theorem 5. *Let I be a proper ideal in the polynomial ring $\mathbb{C}[z_1, z_2, \dots, z_n]$. Then to any irreducible variety V_i of the associated primes corresponding to the primary decomposition of I there exists a nonempty set $\mathcal{P}_i$ of linear differential operators with polynomial coefficients such that a polynomial p belongs to I if and only if*

$$P(\xi, \partial)p(\xi) = 0 \tag{3}$$

holds for each ξ in V_i and for each P in $\mathcal{P}_i$.

Proof. The statement follows from the Lasker–Noether Primary Decomposition Theorem (see e.g. [6]), which says that any proper ideal in the polynomial ring is the intersection of finitely many primary ideals. □

In the case $n = 1$ any proper ideal in $\mathbb{C}[z]$ is a principal ideal, hence its variety V is a nonempty finite set:

$$V = \{\xi_1, \xi_2, \dots, \xi_k\},$$

where the complex numbers $\xi_1, \xi_2, \dots, \xi_k$ are the different roots of the generating polynomial of I with positive multiplicities $m_1, m_2, \dots, m_k$. These numbers, as singletons represent the varieties of the associated primes. In this case $\mathcal{P}_{\xi_j}$ can be taken as the set of differential operators $\{1, D, D^2, \dots, D^{m_j-1}\}$ for $j = 1, 2, \dots, k$, where D is the operator of differentiation. The condition (3) means that a polynomial p belongs to I if and only if its derivatives $p^{(i)}$ for $i = 0, 1, \dots, m_j - 1$ vanish at ξ_j for $j = 1, 2, \dots, k$.

3. Noetherian operators of polynomial ideals

Now we describe the sets of polynomials $\mathcal{P}$ for a given proper ideal I in $\mathbb{C}[z_1, z_2, \dots, z_n]$. We need the following simple result.

Theorem 6. *Let P, f, g be given polynomials in $\mathbb{C}[z_1, z_2, \dots, z_n]$. Then we have*

$$P(\partial)(f \cdot g) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha f \cdot [(\partial^\alpha P)(\partial)] g \,. \tag{4}$$

Proof. The statement is obvious if P is a monomial of the form $P(z) = z^\beta$ by Leibniz's Rule:

$$\partial^\beta (f \cdot g) = \sum_{\alpha \in \mathbb{N}^n} \frac{\beta!}{\alpha!(\beta - \alpha)!} \partial^\alpha f \cdot \partial^{\beta - \alpha} g.$$

Hence the general statement follows. $\square$

Let ξ be a fixed point in the variety V of I and let $\mathcal{P}_\xi(I)$ be the set of all polynomials P in $\mathbb{C}[z_1, z_2, \dots, z_n]$ for which

$$P(\partial) f(\xi) = 0$$

holds for each f in I. Obviously $\mathcal{P}_\xi(I)$ is a linear space of polynomials. By the following theorem it is also *closed under differentiation*: if P belongs to $\mathcal{P}_\xi(I)$ then $\partial^\alpha P$ belongs to $\mathcal{P}_\xi(I)$ for any multi-index α.

Theorem 7. *Let I be a proper ideal in $\mathbb{C}[z_1, z_2, \ldots, z_n]$ and let ξ be a common zero of all polynomials in I. Then the set of all polynomials P in $\mathbb{C}[z_1, z_2, \ldots, z_n]$ which satisfy*

$$P(\partial)f(\xi) = 0 \tag{5}$$

for all f in I is a nonzero translation invariant linear space closed under differentiation. Conversely, if $\mathcal{P}$ is any nonzero linear space of polynomials, which is closed under differentiation then the set of all polynomials f satisfying (5) *with some fixed ξ in $\mathbb{C}^n$ is a proper ideal.*

Proof. Suppose first that the polynomial P satisfies (5) for any f in I. Fix f in I, and let $g(x) = x_1$, then $g \cdot f$ is in I and we have by Theorem 6

$$0 = P(\partial)[g \cdot f](\xi) = \sum_\alpha \frac{1}{\alpha!} \partial^\alpha g(\xi) \big[\partial^\alpha P\big] f(\xi) =$$

$$= \xi \cdot [P(\partial)f](\xi) + [\partial_1 P(\partial)] f(\xi)\,,$$

which implies that $\partial_1 P$ satisfies (5) for any f in I. By iteration we obtain that the given linear space of polynomials is closed under differentiation. It is nonzero, because by Hilbert's Nullstellensatz, it includes all constant polynomials. On the other hand, using the Newton Interpolation Formula and the Taylor Formula we see that a linear space of polynomials is closed under differentiation if and only if it is translation invariant: derivatives are linear combinations of translates and translates are linear combinations of derivatives.

Conversely, suppose now that I is the set of all polynomials f satisfying (5) with some fixed ξ in $\mathbb{C}^n$ for any P from a nonzero linear space of polynomials $\mathcal{P}$, which is closed under differentiation. As $\mathcal{P}$ includes all constant polynomials, hence I is proper. Clearly, I is closed under addition. On the other hand, if f is in I and g is an arbitrary polynomial, then by Theorem 6 we have for any P in $\mathcal{P}$

$$P(\partial)(g\cdot f)(\xi)=\sum_{\alpha}\frac{1}{\alpha}\partial^\alpha g(\xi)[\partial^\alpha P(\partial)]f(\xi)=0\,,$$

as $\mathcal{P}$ is closed under differentiation. Hence $g\cdot f$ is in I and I is an ideal. $\square$

The following theorem gives a complete description of $\mathcal{P}_\xi(I)$.

Theorem 8. *Let I be a proper ideal in $\mathbb{C}[z_1,z_2,\dots,z_n]$ and let ξ be a point in the variety V of I. Then the polynomial P belongs to $\mathcal{P}_\xi(I)$ if and only if*

$$\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,\partial^\alpha f(\xi)\partial^\alpha P(z)=0 \tag{6}$$

holds for each f in I and for all z in $\mathbb{C}^n$.

Proof. Obviously we may suppose that $I\neq\{0\}$. First suppose that P satisfies (6) for each f in I and for each z in $\mathbb{C}^n$. For any multi-index α and for any z in $\mathbb{C}^n$ we let $q_\alpha(z)=(z-\xi)^\alpha$. Then it follows

$$[P(\partial)q_\alpha](\xi)=\partial^\alpha P(0)\,, \tag{7}$$

hence

$$P(\partial)f(\xi)=P(\partial)\big[\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,\partial^\alpha f(\xi)q_\alpha\big](\xi)=$$

$$=\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,\partial^\alpha f(\xi)[P(\partial)q_\alpha](\xi)=\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,\partial^\alpha f(\xi)\partial^\alpha P(0)=0\,,$$

for any f in I by (6). This means that P is in $\mathcal{P}_\xi(I)$.

Conversely, suppose that P is in $\mathcal{P}_\xi(I)$. Then we have as above

$$0=P(\partial)f(\xi)=P(\partial)\big[\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,\partial^\alpha f(\xi)q_\alpha\big](\xi)=$$

$$=\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,\partial^\alpha f(\xi)[P(\partial)q_\alpha](\xi)=\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,\partial^\alpha f(\xi)\partial^\alpha P(0)\,.$$

As $\mathcal{P}_\xi(I)$ is translation invariant, this latter equation holds for any translate of P. Replacing P by $w\mapsto P(w+z)$ our statement follows. $\square$

By Theorem 4 and Theorem 5 Noetherian operators for a given proper ideal I (which are not unique) can be found in the following way. Suppose that $f_1, f_2, \dots, f_N$ is a Groebner–basis for I and the irreducible varieties of the associated primes in the primary decomposition of I are the sets $V_1, V_2, \dots, V_r$ in $\mathbb{C}^n$. We consider the partial differential equations (6) with f_i instead of f for $i = 1, 2, \dots, N$ and with ξ in V_j for $j = 1, 2, \dots, r$. The polynomial solutions $P = P(\xi, z)$ of this system depend polynomially on ξ, as it is shown by (2). These polynomial solutions form the differentiation invariant linear spaces of polynomials for each V_j, independently. Then the linear differential operators with polynomial coefficients $P(\xi, \partial)$ form a set of Noetherian operators. The way, how to choose a finite number of them depends on the special form of the corresponding systems of partial differential equations. If each of them has a finite dimensional solution space, then we simply take bases of them, and since their number r is finite, the problem is solved. If any of them has an infinite dimensional solution space, then in the general polynomial solution arbitrary polynomials appear. More exactly, there are a finite number of polynomials such that any polynomial of them is a solution, too. In this case it turns out, that instead of "any polynomial" we can take a polynomial of degree large enough to obtain a finite number of Noetherian operators.

Here we present a simple example to illustrate the method. The example is taken from [4].

Let I be the ideal generated by the polynomials $xz - y$, y^2 and z^2. Then I is primary to the ideal generated by y and z. The variety of I is obtained by solving the system $xz - y = 0$, $y^2 = 0$, $z^2 = 0$ and the solution set is the x-axis. We write $V = \{(\xi, 0, 0) \,|\, \xi \in \mathbb{C}\}$. For any $(\xi, 0, 0)$ in V we need the Taylor–series of the three generating polynomials at $(\xi, 0, 0)$, by (6). We have

$$xz - y = (x - \xi)z + \xi z - y, \qquad y^2 = y^2, \qquad z^2 = z^2\,,$$

and hence, by replacing $x - \xi$ by ∂_1, y by ∂_2 and z by ∂_3 the system of partial differential equations (6) for the Noetherian operators at $(\xi, 0, 0)$ takes the form

$$\begin{gathered}(\partial_1\partial_3 + \xi\partial_3 - \partial_2)P(x, y, z) = 0\,,\\ \partial_2^2 P(x, y, z) = 0\,,\end{gathered}$$

$$\partial_3^2 P(x,y,z) = 0\,.$$

By a straightforward computation we have that P has the form

$$P(x,y,z) = A(x) + B(x)(\xi y + z) + B'(x)y$$

with arbitrary polynomials A, B. This means, that any Noetherian operator for I has the form

$$P(\partial_1,\partial_2,\partial_3) = A(\partial_1) + B(\partial_1)(\xi\partial_2 + \partial_3) + B'(\partial_1)\partial_2 \tag{8}$$

with arbitrary polynomials A, B. However, it is obvious that if the two operators 1 and $\xi\partial_2 + \partial_3$ annihilate a polynomial f at $(\xi, 0, 0)$, then so does any operator of the form (8). Hence we can take these two operators as Noetherian operators for I, which means that a polynomial f belongs to I if and only if it satisfies

$$f(x,0,0) = 0, \qquad x\partial_2 f(x,0,0) + \partial_3 f(x,0,0) = 0$$

for any complex number x.

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