NON-SYNTHESIZABLE VARIETIES

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Abstract. In 2004 a counterexample was given for a 1965 result of R. J. Elliott claiming that discrete spectral synthesis holds on every Abelian group. Here we present a ring-theoretical approach to this problem, and show that some varieties fail to have spectral synthesis. In particular, we give a new proof for the result of the second author that spectral synthesis does not hold on Abelian groups with infinite torsion free rank.

1. Introduction

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. One considers the space $\mathcal{C}(G)$ of all complex valued continuous functions on a locally compact Abelian group G, which is a locally convex topological linear space with respect to point-wise linear operations (addition, multiplication with scalars) and to the topology of uniform convergence on compact sets. The *translate* by y in G of an element f in $\mathcal{C}(G)$ is defined by $\tau_y f(x) = f(x+y)$ for each x in G. A subset in $\mathcal{C}(G)$ is called *translation invariant* if it contains every translates of all of its elements. A closed translation invariant linear subspace of the space $\mathcal{C}(G)$ is called a *variety* on G. Continuous homomorphisms of G into the additive [multiplicative] topological group of [nonzero] complex numbers are called *additive [exponential] functions*. A function is a *polynomial* if it belongs to the algebra generated by the additive functions and constants. Usually, the product of a polynomial and an exponential is called an *exponential monomial*. This is equivalent to the property that the function generates a finite dimensional indecomposable variety (see e.g. [11]). We shall use this latter definition here.

It turns out that exponential functions, or more generally, exponential monomials can be considered as basic building blocks of varieties. A given variety may or may not contain any exponential function or exponential monomial. If each nonzero subvariety of a given variety contains an exponential function, then we say that *spectral analysis* holds for the variety. Another property is if the variety is *synthesizable*, which means that all exponential monomials in this variety span a dense subspace in the variety. If each subvariety of a given variety is synthesizable, then we say that *spectral synthesis* holds for the variety. It can be shown that spectral synthesis for a variety implies spectral analysis, too (see [11], Theorem 1). If spectral analysis, respectively, spectral synthesis holds for every nonzero variety on an Abelian group, then we say that *spectral analysis*,

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respectively, *spectral synthesis holds on the group*. A famous and pioneer result of L. Schwartz [8] exhibits the situation in a classical case by stating that if the underlying group is the reals with the Euclidean topology, then every nonzero variety contains an exponential function, that is, spectral analysis holds on the reals. Moreover, spectral synthesis also holds: there are sufficiently many exponential monomials in each variety in the sense that their linear hull is dense in the variety.

In his 1958 result [7] M. Lefranc proved that spectral synthesis holds on the group $\mathbb{Z}^n$ for each positive integer n, $\mathbb{Z}$ being the integers. In his 1965 paper [1] R. J. Elliot presented a theorem on spectral synthesis for arbitrary Abelian groups. However, in his 1987 private communication [2] Z. Gajda called the second author's attention to the fact that the proof of Elliot's theorem had several gaps. Later on several efforts have been made to solve the problem of discrete spectral analysis and spectral synthesis on arbitrary Abelian groups. Finally, a counterexample for Elliot's theorem was presented at the 41st International Symposium on Functional Equations, Noszvaj, Hungary, 2003 (see [9]). For basics, further developments and references on spectral analysis and spectral synthesis the reader should refer to [3, 10, 11].

In this paper we give a new proof for the failure of spectral synthesis on some types of discrete Abelian groups, which was shown by a counterexample in [9]. Our method is based on ring-theoretical results and uses the annihilator technique. The basics of this method have been worked out in [12].

2. Basic concepts

In this paper we consider discrete commutative groups only, and $\mathcal{C}(G)$ is the set of all functions from G to $\mathbb{C}$. Let G be an Abelian group and let $\mathbb{C}G$ denote its *group algebra*, which is identified with the set of all finitely supported complex valued functions on G. Moreover, this set can be identified with the set of all finitely supported complex measures $\mathcal{M}_c(G)$ on G, using the definition

$$\int_G f\,d\mu = \sum_{x\in G} f(x)\mu(x)\,,$$

whenever μ is in $\mathbb{C}G$ and f is in $\mathcal{C}(G)$. This formula expresses the well-known fact about the dual $\mathcal{C}(G)^*$ of the topological vector space $\mathcal{C}(G)$: it is identified with $\mathcal{M}_c(G)$, the pairing given by the previous formula.

Via these identifications the multiplication in the complex algebra $\mathbb{C}G$ is given by the *convolution* of measures as

$$\mu * \nu(f) = \sum_{y\in G} f(x+y)\mu(x)\nu(y)$$

for each μ, ν in $\mathbb{C}G$ and f in $\mathcal{C}(G)$. With this operation $\mathbb{C}G$ is a commutative unital complex algebra with identity δ_0, which is the point mass concentrated at 0, the zero element of G. More generally, we denote by δ_x the characteristic function of the singleton $\{x\}$ for each x in G: it takes the value 1 at the element x and 0 otherwise.

Convolution is also defined between elements of $\mathbb{C}G$ and $\mathcal{C}(G)$ in the following manner:

$$\mu * f(x) = \sum_{y\in G} f(x-y)\mu(y)\,,$$

whenever μ is in $\mathbb{C}G$, f is in $\mathcal{C}(G)$ and x is in G. With this operation $\mathcal{C}(G)$ turns into a $\mathbb{C}G$-module, closed submodules being exactly the varieties. The intersection of all varieties including a particular f in $\mathcal{C}(G)$ is called the *variety of f* and is denoted by $\tau(f)$.

For each subset H in $\mathcal{C}(G)$ the *annihilator* $H^\perp$ of H in $\mathbb{C}G$ is defined by

$$H^\perp = \{\mu :\, \mu * f = 0 \text{ for each } f \text{ in } H\}\,.$$

It is easy to see that $H^\perp$ is an ideal in $\mathbb{C}G$. If $H = \{f\}$ is a singleton, then $H^\perp = \tau(f)^\perp$ and we call it the *annihilator of f*.

Similarly, the *annihilator* $K^\perp$ in $\mathcal{C}(G)$ of a subset K in $\mathbb{C}G$ is defined by

$$K^\perp = \{f :\, \mu * f = 0 \text{ for each } \mu \text{ in } K\}\,.$$

It is also easy to check that $K^\perp$ is a variety in $\mathcal{C}(G)$.

The following two theorems are important technical tools (see [6, 12]).

Theorem 2.1. *Let G be an Abelian group, V a variety on G and I an ideal in $\mathbb{C}G$. Then we have*

$$V^{\perp\perp} = V,\;\; I^{\perp\perp} = I\,.$$

Theorem 2.2. *Let G be an Abelian group, $(V_\gamma)_{\gamma\in\Gamma}$ a family of varieties on G and $(I_\gamma)_{\gamma\in\Gamma}$ a family of ideals in $\mathbb{C}G$. Then we have*

$$(\sum_{\gamma\in\Gamma} V_\gamma)^\perp = \bigcap_{\gamma\in\Gamma} V_\gamma^\perp,\;\; (\bigcap_{\gamma\in\Gamma} I_\gamma)^\perp = \sum_{\gamma\in\Gamma} I_\gamma^\perp\,.$$

3. Exponentials and maximal ideals

The basic building blocks of spectral analysis and spectral synthesis are exponential monomials. We call the reader's attention that we shall use the word "exponential" in several different meanings in the sequel. The generalized characters of G, that is, the homomorphisms of G into the multiplicative group of nonzero complex numbers will be called *exponential functions*, or simply *exponentials*. Later on we shall also use the terms "exponential maximal ideal", "exponential monomial", and "generalized exponential monomial", which refer to different, however, related concepts.

Exponentials can be characterized by a number of properties. We shall use the following result (see [12, Theorems 3 and 4], and [12, Corollaries 1 and 2]).

Theorem 3.1. *Let G be an Abelian group and $m\colon G\to\mathbb{C}$ be an arbitrary function. Then the following conditions are equivalent:*

(1) *m is an exponential.*
(2) *The variety of m is one dimensional and $m(0) = 1$.*
(3) *The annihilator $\tau(m)^{\perp}$ is a maximal ideal in $\mathbb{C}G$, $\mathbb{C}G/\tau(m)^{\perp}$ is isomorphic to $\mathbb{C}$ and $m(0) = 1$.*

Maximal ideals M in $\mathbb{C}G$ with the property that $\mathbb{C}G/M \cong \mathbb{C}$ play an important role, and they will be called *exponential maximal ideals*. They are closely related to modified differences defined as follows. For each function $f\colon G \to \mathbb{C}$ and y in G we define

$$\Delta_{f;y} = \delta_{-y} - f(y)\delta_0 \,.$$

The measure $\Delta_{f;y}$ is called *modified difference*. For products of modified differences we shall use the notation

$$\Delta_{f;y_1,y_2,\dots,y_{n+1}} = \Pi_{i=1}^{n+1}\Delta_{f;y_i} \,,$$

whenever $y_1, y_2, \dots, y_{n+1}$ are in G. The product on the right side is meant as a convolution.

Given f in $\mathcal{C}(G)$ the ideal in $\mathbb{C}G$ generated by all modified differences of the form $\Delta_{f;y}$ with y in G is denoted by M_f. It is reasonable to ask whether M_f is proper. We have the following result.

Theorem 3.2. *Let G be an Abelian group and $f\colon G \to \mathbb{C}$ be a function. The ideal M_f is proper if and only if f is an exponential. In this case $M_f = \tau(f)^{\perp}$, hence M_f is an exponential maximal ideal.*

Proof. Suppose first that M_f is proper. Then $M_f^{\perp}$ is a nonzero variety, by Theorem 2.1, hence there is a nonzero g annihilated by all modified differences of the form $\Delta_{f;y}$. For x, y in G we have

$$0 = \Delta_{f;y} * g(x) = g(x+y) - f(y)g(x) \,. \tag{3.1}$$

Putting $x = 0$ we have $g(y) = g(0)\cdot f(y)$. In particular, $g(0) \neq 0$, $f \neq 0$ and we obtain $f(x+y) = f(x)f(y)$. As f is nonzero, it follows $f(0) = 1$, hence f is an exponential.

Conversely, suppose that $f = m$ is an exponential. Then m is in $M_m^{\perp}$, as obviously $\Delta_{m;y} * m(x) = m(x+y) - m(y)m(x) = 0$ for each x, y in G. Hence M_m is proper. Moreover, $\tau(m)^{\perp}$ is an exponential maximal ideal in $\mathbb{C}G$, by Theorem 3.1. If g is in $M_f^{\perp}$, then, by (3.1), it is a constant multiple of m, hence it belongs to $\tau(m)$. It follows $M_m^{\perp} \subseteq \tau(m)$, thus $\tau(m)^{\perp} \subseteq M_m$. As $\tau(m)^{\perp}$ is maximal and M_m is proper, we have $\tau(m)^{\perp} = M_m$, and the theorem is proved. □

These latter two results have been used in [12] to prove the following characterization results.

Theorem 3.3. *Let G be an Abelian group and V be a variety on G. Then spectral analysis holds for V if and only if each maximal ideal containing V is exponential.*

Corollary 3.4. *Let G be an Abelian group and V be a variety on G. Then spectral analysis holds on G if and only if each maximal ideal in $\mathbb{C}G$ is exponential.*

Corollary 3.5. *Let G be an Abelian group and V be a variety on G. Then spectral analysis holds for V if and only if each maximal ideal in $\mathbb{C}G/V^\perp$ is exponential.*

4. Exponential monomials

Let G be an Abelian group. The variety V on G is called *decomposable*, if it is the sum of two proper subvarieties, which means that the algebraic sum of two proper subvarieties is a dense submodule in it. Otherwise it is called *indecomposable*. The following theorem is obvious, by Theorem 2.2.

Theorem 4.1. *Let G be an Abelian group. A variety on G is decomposable if and only if its annihilator is the intersection of two ideals, which are different from it.*

Let G be an Abelian group. The function $f\colon G \to \mathbb{C}$ is called a *generalized exponential monomial*, if there exists an exponential m and a natural number n such that for each $y_1, y_2, \dots, y_{n+1}$ we have

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f = 0 \tag{4.1}$$

holds. We reformulate this definition in terms of the annihilator of f.

Theorem 4.2. *Let G be an Abelian group. The function $f\colon G \to \mathbb{C}$ is a generalized exponential monomial if and only if its annihilator contains some positive power of an exponential maximal ideal.*

Proof. The condition of the theorem is equivalent to the following condition: there exists an exponential m and a natural number n such that

$$M_m^{n+1} \subseteq \tau(f)^\perp\,. \tag{4.2}$$

As the modified differences $\Delta_{m;y}$ with y in G generate M_m, hence the modified differences $\Delta_{m;y_1,y_2,\dots,y_{n+1}}$ generate M_m^{n+1}, that is, (4.1) and (4.2) are equivalent for f. ☐

It is easy to check (see [12, Theorem 7]) that condition (4.2) can hold for at most one exponential m.

Theorem 4.3. *Let G be an Abelian group and $f\colon G \to \mathbb{C}$ be a nonzero generalized exponential monomial. Then there is a unique exponential m satisfying (4.1) for some natural number n. In other words, there is a unique exponential maximal ideal M satisfying $M^{n+1} \subseteq \tau(f)^\perp$ for some natural number n.*

Now we have the following characterization result (see [12, Theorem 8]).

Theorem 4.4. *Let G be an Abelian group. The function $f\colon G \to \mathbb{C}$ is a nonzero generalized exponential monomial if and only if $\mathbb{C}G/\tau(f)^\perp$ is a local ring with nilpotent exponential maximal ideal.*

Theorem 4.5. *Let G be an Abelian group and $f\colon G \to \mathbb{C}$ be an exponential monomial. Then $\mathbb{C}G/\tau(f)^\perp$ is a local Artin ring.*

Proof. By the previous theorem $\mathbb{C}G/\tau(f)^\perp$ is a local ring. Any descending chain of ideals in $\mathbb{C}G/\tau(f)^\perp$ induces a descending chain of ideals containing $\tau(f)^\perp$ in $\mathbb{C}G$, which induces an ascending chain of subvarieties in $\tau(f)$, hence, by finite dimensionality, it terminates. □

5. The failure of spectral synthesis

Theorem 5.1. *Let G be an Abelian group and V be a variety on G. If the variety V is synthesizable, then $\mathbb{C}G/V^\perp$ can be embedded into a direct product of local Artin rings.*

Proof. If V is synthesizable, then it is the topological sum of all subvarieties generated by exponential monomials belonging to V, by definition. This means that we have, by Theorem 2.2,

$$V^\perp = \bigcap_{\varphi \in V} \tau(\varphi)^\perp\,, \tag{5.1}$$

where the intersection is extended to all exponential monomials φ in V. We define the mapping

$$F\colon \mathbb{C}G \to \Pi_\varphi \mathbb{C}G/\tau(\varphi)^\perp$$

by

$$F(\mu)(\varphi) = \mu + \tau(\varphi)^\perp$$

for each exponential monomial φ in V. Then, F is a ring homomorphism of $\mathbb{C}G$ into the direct product $\Pi_\varphi \mathbb{C}G/\tau(\varphi)^\perp$. The kernel of F consists of those μ in $\mathbb{C}G$ belonging to $\tau(\varphi)^\perp$ for each exponential monomial φ in V, that is, by (5.1), the kernel of F is $V^\perp$. It follows that $\mathbb{C}G/V^\perp$ is isomorphic to its image by F, which is a subring of the direct product of local Artin rings, by Theorem 4.5. □

We need the following lemma.

Lemma 5.2. *Let $(R_\gamma)_{\gamma\in\Gamma}$ be an arbitrary family of (unital) local rings and I be an ideal in the direct product $R = \Pi_{\gamma\in\Gamma} R_\gamma$. If I is a (unital) local ring, then there exists γ_0 in Γ such that $I = R_{\gamma_0}$.*

Proof. Let $e = (e_\gamma)_{\gamma\in\Gamma}$ be the identity element of I. Then e_γ is an idempotent for each γ in Γ. Now, a unital local ring contains only two idempotents: the identity element and 0. Indeed, if a is idempotent in a local ring, then a is either a unit (and then $a^2 = a$ yields $a = 1$) or an element of the Jacobson radical. In the latter case, if S is any subring of the Jacobson radical, then $a^2 S = aS$, hence $a = 0$, by [5, Theorem 6]. That is, either $e_\gamma = 0_\gamma$ or $e_\gamma = 1_\gamma$ for every γ in Γ. Let $E \subseteq \Gamma$ be the set such that $e_\gamma = 1_\gamma$ if and only if γ is in E. It follows $I = ReR = \prod_{\gamma\in E} R_\gamma$. Since I is local, it is direct indecomposable. Thus, E contains only one element γ_0, and $I = R_{\gamma_0}$. □

Theorem 5.3. *Let G be an Abelian group and $m\colon G \to \mathbb{C}$ be an exponential. If $(M_m^2)^\perp$ is infinite dimensional, then it is non-synthesizable.*

Proof. Now, $\mathbb{C}G/M_m^2$ is a local ring, as the only maximal ideal containing M_m^2 is M_m. Indeed: assume that M is a maximal ideal containing M_m^2. Then M is a prime ideal, and therefore $M_m \subseteq M$. As M_m is maximal, we have $M = M_m$.

For each φ in $(M_m^2)^\perp$ and μ in $\mathbb{C}G$ we define

$$(\mu + M_m) * (\varphi + M_m^\perp) = \mu * \varphi + M_m^\perp \,. \tag{5.2}$$

We show that this definition turns the $\mathbb{C}G$-module $(M_m^2)^\perp/M_m^\perp$ into a $\mathbb{C}G/M_m$-module. Indeed, if $\mu - \tilde{\mu}$ is in M_m and $\varphi - \tilde{\varphi}$ is in $M_m^\perp$, then

$$\mu * \varphi - \tilde{\mu} * \tilde{\varphi} = \mu * \varphi - \tilde{\mu} * \varphi + \tilde{\mu} * \varphi - \tilde{\mu} * \tilde{\varphi} = (\mu - \tilde{\mu}) * \varphi + \tilde{\mu} * (\varphi - \tilde{\varphi}) \,.$$

As φ is in $(M_m^2)^\perp$ and $\mu - \tilde{\mu}$ is in M_m it follows that the first term on the right side is in $M_m^\perp$. On the other hand, as $\varphi - \tilde{\varphi}$ is in $M_m^\perp$, which is a $\mathbb{C}G$-module, hence also the second term is in $M_m^\perp$. Consequently, the definition given by (5.2) uniquely defines the action of $\mathbb{C}G/M_m$ on $(M_m^2)^\perp/M_m^\perp$, and it is easy to check that, in fact, $(M_m^2)^\perp/M_m^\perp$ is a $\mathbb{C}G/M_m$-module. Let $F\colon \mathbb{C}G \to \mathbb{C}G/M_m$ denote the natural homomorphism, where we identify $\mathbb{C}G/M_m$ with $\mathbb{C}$. For each y in G and φ in $(M_m^2)^\perp$ we have

$$\delta_{-y} * \varphi + M_m^\perp = \big(\delta_{-y} - m(y)\delta_0\big) * \varphi + m(y)\varphi + M_m^\perp =$$

$$m(y)\varphi + M_m^\perp = F(\delta_{-y})\varphi + M_m^\perp = F(\delta_{-y})(\varphi + M_m^\perp) \,,$$

as $\delta_{-y} - m(y)\delta_0$ is in M_m, hence the first term on the right side is in $M_m^\perp$. As every μ in $\mathbb{C}G$ is a linear combination of a finite number of δ_y's, we conclude that

$$\mu * (\varphi + M_m^\perp) = \mu * \varphi + M_m^\perp = F(\mu)\varphi + M_m^\perp = F(\mu)(\varphi + M_m^\perp)$$

holds for each μ in $\mathbb{C}G$. In other words, $\mathbb{C}G * (\varphi + M_m^\perp) = \mathbb{C} \cdot (\varphi + M_m^\perp)$, that is, for each φ in $(M_m^2)^\perp$ the element $\varphi + M_m^\perp$ generates a one dimensional submodule in the $\mathbb{C}G$-module $(M_m^2)^\perp/M_m^\perp$. This implies that given the functions $\varphi_1, \varphi_2, \dots, \varphi_n$ in $(M_m^2)^\perp$ the submodule generated by the elements $\varphi_j + M_m^\perp$ for $j = 1, 2, \dots, n$ in the $\mathbb{C}G$-module $(M_m^2)^\perp/M_m^\perp$ coincides with the linear span of these elements.

By assumption, $(M_m^2)^\perp$ is infinite dimensional. As $M_m^\perp = \tau(m)$ is one dimensional, hence $(M_m^2)^\perp/M_m^\perp$ is infinite dimensional. Let $(\varphi_j)_{j\in\mathbb{N}}$ be a sequence of functions in $(M_m^2)^\perp$ such that the elements $\varphi_j + M_m^\perp$ are linearly independent. Then $\varphi_{n+1} + M_m^\perp$ is not included in the submodule in $(M_m^2)^\perp/M_m^\perp$ generated by the elements $\varphi_j + M_m^\perp$ for $0 \le j \le n$ $(n = 0, 1, \dots)$. It follows, that φ_{n+1} is not included in the variety V_n in $(M_m^2)^\perp$ generated by the elements φ_j for $0 \le j \le n$ $(n = 0, 1, \dots)$. We infer that the varieties V_n form a strictly ascending chain, hence their annihilators $V_n^\perp$ form a strictly descending chain of ideals containing M_m^2, which induces a strictly descending chain of ideals in the ring $\mathbb{C}G/M_m^2$. Consequently, the ring $\mathbb{C}G/M_m^2$ it is not an Artin ring. By Theorem 5.1, if $(M_m^2)^\perp$ is synthesizable, then $\mathbb{C}G/M_m^2$ can be embedded into a direct product of local Artin rings. As $\mathbb{C}G/M_m^2$ is local, then its image at this embedding must be equal to one of the factors of the direct product, by Lemma 5.2, which is impossible. The theorem is proved. ☐

The following theorem can be proved exactly in the same way.

Theorem 5.4. *Let G be an Abelian group, V be a variety on G and $m\colon G\to\mathbb{C}$ be an exponential. If $(M_m^2)^\perp\cap V$ is infinite dimensional, then V is non-synthesizable.*

Theorem 5.5. *Let G be an Abelian group with infinite torsion free rank. Then spectral synthesis fails to hold on G.*

Proof. Let $H=\bigoplus_{n\in\mathbb{N}} H_i$, where $H_n=\mathbb{Z}$ for each n in $\mathbb{N}$. In other words, H is the set of all finitely supported integer valued functions on $\mathbb{N}$. With the pointwise addition H is an Abelian group. As G has infinite torsion free rank, it has a subgroup isomorphic to H. Without loss of generality we may suppose that H is a subgroup of G. For each n in $\mathbb{N}$ let $p_n\colon H\to\mathbb{Z}$ denote the n-th projection. Clearly, for each n in $\mathbb{N}$ the function p_n is a homomorphism of H into $\mathbb{Z}$, hence, by Theorem A.7 in [4], p. 441, each p_n has an extension to a homomorphism $\tilde{p}_n$ of G into $\mathbb{Z}$, and these extensions are linearly independent, too. Obviously, for each x,y,z in G we have

$$\Delta_{1;y,z}*\tilde{p}_n(x)=\tilde{p}_n(x+y+z)-\tilde{p}_n(x+y)-\tilde{p}_n(x+z)+\tilde{p}_n(x)=0\,,$$

that is, $\tilde{p}_n$ is in $(M_1^2)^\perp$, which implies that $(M_1^2)^\perp$ is infinite dimensional. By the previous theorem, $(M_1^2)^\perp$ is non-synthesizable and our theorem is proved. □

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